Classification of quasifinite representations with nonzero central charges for type A_1 EALA with coordinates in quantum torus*

Weiqiang Lin[1,2]   Yucai Su[2]

1. Department of Mathematics, Zhangzhou Teachers College,
Zhangzhou 363000, Fujian, China

2. Department of Mathematics, University of Science and Technology of China,
Hefei 230026, Anhui, China

E-mail: linwq83@yahoo.com.cn, ycsu@ustc.edu.cn

Abstract: In this paper, we first construct a Lie algebra L from rank 3 quantum torus, and show that it is isomorphic to the core of EALAs of type A_1 with coordinates in rank 2 quantum torus. Then we construct two classes of irreducible $\mathbf{Z}$-graded highest weight representations, and give the necessary and sufficient conditions for these representations to be quasifinite. Next, we prove that they exhaust all the generalized highest weight irreducible $\mathbf{Z}$-graded quasifinite representations. As a consequence, we determine all the irreducible $\mathbf{Z}$-graded quasifinite representations with nonzero central charges. Finally, we construct two classes of highest weight $\mathbf{Z}^2$-graded quasifinite representations by using these $\mathbf{Z}$-graded modules.

Keyword: core of EALAs, graded representations, quasifinite representations, highest weight representations, quantum torus.

§1 Introduction

Extended affine Lie algebras (EALAs) are higher dimensional generalizations of affine Kac-Moody Lie algebras introduced in [1] (under the name of irreducible quasi-simple Lie algebras). They can be roughly described as complex Lie algebras which have a nondegenerate invariant form, a self-centralizing finite-dimensional ad-diagonalizable Abelian subalgebra (i.e., a Cartan subalgebra), a discrete irreducible root system, and ad-nilpotency of nonisotropic root spaces (see [2–4]). Toroidal Lie algebras, which are universal central extensions of $\dot{\mathfrak{g}} \otimes \mathbf{C}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}]$ ($\dot{\mathfrak{g}}$ is a finite-dimensional simple Lie algebra), are prime examples of EALAs studied in [5–11], among others. There are many EALAs which allow not only Laurent polynomial algebra $\mathbf{C}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}]$ as coordinate algebra but also quantum tori, Jordan tori and the octonians tori as coordinated algebras depending on type of the Lie algebra (see [2, 3, 12–14]). The structure theory of the EALAs of type A_{d-1} is tied up with Lie algebra $gl_d(\mathbf{C}) \otimes \mathbf{C}_Q$ where $\mathbf{C}_Q$ is the quantum torus. Quantum torus defined in [15] are noncommutative analogue of Laurent polynomial algebras.

The universal center extension of the derivation Lie algebra of rank 2 quantum torus is known as the q-analog Virasoro-like algebra (see [16]). Representations for Lie algebras coordinated by certain quantum tori have been studied by many people (see [17–22] and the references therein). The structure and representations of the q-analog Virasoro-like algebra are studied in many papers (see [23–27]). In this paper, we first construct a Lie algebra L from rank 3 quantum torus, which contains the q-analog Virasoro-like algebra as its Lie subalgebra, and show that it is isomorphic to the core of EALAs of type A_1 with coordinates in rank 2 quantum torus. Then we study quasifinite representation of L.

When we study quasifinite representations of a Lie algebra of this kind, as pointed out by Kac and Radul in [28], we encounter the difficulty that though it is $\mathbf{Z}$-graded, the graded subspaces are still infinite dimensional, thus the study of quasifinite modules is a nontrivial problem.

Now we explain this paper in detail. In Section 2, we first recall some concepts about the quantum torus and EALAs of type A_1. Next, we construct a Lie algebra L from a special class of rank 3 quantum, and show that L is isomorphic to the core of EALAs of type A_1 with coordinates in rank 2 quantum torus. Then, we prove some basic propositions and reduce the classification of irreducible $\mathbf{Z}$-graded representations of L to that of the generalized highest weight representations and the uniformly bounded representations. In Section 3, we construct two class of irreducible $\mathbf{Z}$-graded highest weight representations of L, and give the necessary and sufficient conditions for these representations to be quasifinite. In Section 4, we prove that the generalized highest weight irreducible $\mathbf{Z}$-graded quasifinite representations of L must be the highest weight representations, and thus the representations constructed in Section 3 exhaust all the generalized highest weight quasifinite representations. As a consequence, we complete the classification of irreducible $\mathbf{Z}$-graded quasifinite representations of L with nonzero central charges, see Theorem 4.4 (the Main Theorem). In Section 5, we construct two classes of highest weight $\mathbf{Z}^2$-graded quasifinite representations.

§2 Basics

Throughout this paper we use $\mathbf{C}, \mathbf{Z}, \mathbf{Z}_+, \mathbf{N}$ to denote the sets of complex numbers, integers, nonnegative integers, positive integers respectively. And we use $\mathbf{C}^*, \mathbf{Z}^{2*}$ to denote the set of nonzero complex numbers and $\mathbf{Z}^2 \setminus \{(0,0)\}$ respectively. All spaces considered in this paper are over $\mathbf{C}$. As usual, if $u_1, u_2, \cdots, u_k$ are elements on some vector space, we use $\langle u_1, \cdots, u_k \rangle$ to denote their linear span over $\mathbf{C}$. Let q be a nonzero complex number. We shall fix a generic q throughout this paper.

Now we recall the concept of quantum torus from [15]. Let ν be a positive integer and $Q = (q_{ij})$ be a $\nu \times \nu$ matrix, where

$$q_{ij} \in \mathbf{C}^*, \ q_{ii} = 1, \ q_{ij} = q_{ji}^{-1}, \quad \text{for } 0 \leq i,j \leq \nu - 1.$$

A quantum torus associated to Q is the unital associative algebra $\mathbf{C}_Q[t_0^{\pm 1}, \cdots, t_{\nu-1}^{\pm 1}]$ (or, simply

$\mathbf{C}_Q$) with generators $t_0^{\pm 1}, \cdots, t_{\nu-1}^{\pm 1}$ and relations

$$t_i t_i^{-1} = t_i^{-1} t_i = 1 \text{ and } t_i t_j = q_{ij} t_j t_i, \quad \forall \, 0 \le i, j \le \nu - 1.$$

Write $t^{\mathbf{m}} = t_0^{m_0} t_1^{m_1} \cdots t_{\nu-1}^{m_{\nu-1}}$ for $\mathbf{m} = (m_0, m_1, \cdots, m_{\nu-1})$. Then

$$t^{\mathbf{m}} \cdot t^{\mathbf{n}} = \Big(\prod_{0 \le j \le i \le \nu-1} q_{ij}^{m_i n_j} \Big) t^{\mathbf{m}+\mathbf{n}},$$

where $\mathbf{m}, \mathbf{n} \in \mathbf{Z}^\nu$. If $Q = \begin{pmatrix} 1 & q^{-1} \\ q & 1 \end{pmatrix}$, we will simply denote $\mathbf{C}_Q$ by $\mathbf{C}_q$.

Next we recall the construction of EALAs of type A_1 with coordinates in $\mathbf{C}_{q^2}$. Let E_{ij} be the 2×2 matrix which is 1 in the (i,j)-entry and 0 everywhere else. The Lie algebra $\widetilde{\tau} = gl_2(\mathbf{C}_{q^2})$ is defined by

$$[E_{ij}(t^{\mathbf{m}}), E_{kl}(t^{\mathbf{n}})]_0 = \delta_{j,k} q^{2m_2 n_1} E_{il}(t^{\mathbf{m}+\mathbf{n}}) - \delta_{l,i} q^{2n_2 m_1} E_{kj}(t^{\mathbf{m}+\mathbf{n}}),$$

where $1 \le i, j, k, l \le 2, \mathbf{m} = (m_1, m_2)$ and $\mathbf{n} = (n_1, n_2)$ are in $\mathbf{Z}^2$. Thus the derived Lie subalgebra of $\widetilde{\tau}$ is $\overline{\tau} = sl_2(\mathbf{C}_{q^2}) \oplus \langle I(t^{\mathbf{m}}) \mid \mathbf{m} \in \mathbf{Z}^{2*} \rangle$, where $I = E_{11} + E_{22}$, since q is generic. And the universal central extension of $\overline{\tau}$ is $\tau = \overline{\tau} \oplus \langle K_1, K_2 \rangle$ with the following Lie bracket

$$[X(t^{\mathbf{m}}), Y(t^{\mathbf{n}})] = [X(t^{\mathbf{m}}), Y(t^{\mathbf{n}})]_0 + \delta_{\mathbf{m}+\mathbf{n},0} q^{2m_2 n_1} (X, Y)(m_1 K_1 + m_2 K_2),$$

$$K_1, K_2 \text{ are central,}$$

where $X(t^{\mathbf{m}}), Y(t^{\mathbf{n}}) \in \overline{\tau}$ and (X, Y) is the trace of XY. The Lie algebra τ is the core of the EALAs of type A_1 with coordinates in $\mathbf{C}_{q^2}$. If we add degree derivations d_1, d_2 to τ, then $\tau \oplus \langle d_1, d_2 \rangle$ becomes an EALAs since q is generic.

Now we construct our Lie algebra. Let

$$Q = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & q^{-1} \\ 1 & q & 1 \end{pmatrix}.$$

Let J be the two-sided ideal of $\mathbf{C}_Q$ generated by $t_0^2 - 1$. Define

$$\widetilde{L} = \mathbf{C}_Q / J = \langle t_0^i t_1^j t_2^k \mid i \in \mathbf{Z}_2, \ j, k \in \mathbf{Z} \rangle,$$

be the quotient of $\mathbf{C}_Q$ by J and identify t_0 with its image in $\widetilde{L}$. Then the derived Lie subalgebra of $\widetilde{L}$ is $\overline{L} = \langle t_0^{\overline{0}} t^{\mathbf{m}} \mid \mathbf{m} \in \mathbf{Z}^{2*} \rangle \oplus \langle t_0^{\overline{1}} t^{\mathbf{m}} \mid \mathbf{m} \in \mathbf{Z}^2 \rangle$. Now we define a central extension of $\overline{L}$, which will be denoted by $L = \overline{L} \oplus \langle c_1, c_2 \rangle$, with the following Lie bracket

$$[t_0^i t^{\mathbf{m}}, t_0^j t^{\mathbf{n}}] = ((-1)^{m_1 j} q^{m_2 n_1} - (-1)^{i n_1} q^{m_1 n_2}) t_0^{i+j} t^{\mathbf{m}+\mathbf{n}} + (-1)^{m_1 j} q^{m_2 n_1} \delta_{i+j,\overline{0}} \delta_{\mathbf{m}+\mathbf{n},0}(m_1 c_1 + m_2 c_2),$$

3

$$c_1, c_2 \text{ are central,}$$

where $i, j \in \mathbf{Z}_2$, $\mathbf{m} = (m_1, m_2)$ and $\mathbf{n} = (n_1, n_2)$ are in $\mathbf{Z}^2$. One can easily see that $\langle t_0^{\bar{0}} t^{\mathbf{m}} \mid \mathbf{m} \in \mathbf{Z}^{2*}\rangle \oplus \langle c_1, c_2 \rangle$ is a Lie subalgebra of L, which is isomorphic to the q-analog Virasoro-like algebra.

First we prove that the Lie algebra L is in fact isomorphic to the core of the EALAs of type A_1 with coordinates in $\mathbf{C}_{q^2}$.

Proposition 2.1 *The Lie algebra L is isomorphic to τ and the isomorphism is given by the linear extension of the following map φ:*

$$t_0^i t_1^{2m_1+1} t_2^{m_2} \mapsto (-1)^i q^{-m_2} E_{12}(t_1^{m_1} t_2^{m_2}) + E_{21}(t_1^{m_1+1} t_2^{m_2}),$$

$$t_0^i t_1^{2m_1} t_2^{m_2} \mapsto (-1)^i E_{11}(t_1^{m_1} t_2^{m_2}) + q^{-m_2} E_{22}(t_1^{m_1} t_2^{m_2}) + \delta_{i,\bar{1}} \delta_{m_1,0} \delta_{m_2,0} \frac{1}{2} K_1,$$

$$c_1 \mapsto K_1, \qquad\qquad c_2 \mapsto 2K_2,$$

where $t_0^i t_1^{2m_1+1} t_2^{m_2}$, $t_0^i t_1^{2m_1} t_2^{m_2} \in L$.

Proof We need to prove that φ preserves Lie bracket. First we have

$$[(-1)^i q^{-m_2} E_{12}(t_1^{m_1} t_2^{m_2}) + E_{21}(t_1^{m_1+1} t_2^{m_2}), (-1)^j q^{-n_2} E_{12}(t_1^{n_1} t_2^{n_2}) + E_{21}(t_1^{n_1+1} t_2^{n_2})]$$

$$= \left((-1)^j q^{m_2(2n_1+1)} - (-1)^i q^{n_2(2m_1+1)}\right)\left((-1)^{i+j} E_{11}(t_1^{m_1+n_1+1} t_2^{m_2+n_2})\right.$$

$$\left. + q^{-m_2-n_2} E_{22}(t_1^{m_1+n_1+1} t_2^{m_2+n_2})\right)$$

$$+ \delta_{m_1+n_1+1,0} \delta_{m_2+n_2,0} (-1)^j q^{m_2(2n_1+1)}\left((-1)^{i+j}(m_1 K_1 + m_2 K_2) + (m_1+1)K_1 + m_2 K_2\right)$$

$$= \left((-1)^j q^{m_2(2n_1+1)} - (-1)^i q^{n_2(2m_1+1)}\right)\left((-1)^{i+j} E_{11}(t_1^{m_1+n_1+1} t_2^{m_2+n_2})\right.$$

$$\left. + q^{-m_2-n_2} E_{22}(t_1^{m_1+n_1+1} t_2^{m_2+n_2})\right)$$

$$+ \delta_{i+j,\bar{0}} \delta_{m_1+n_1+1,0} \delta_{m_2+n_2,0}(-1)^j q^{m_2(2n_1+1)}((2m_1+1)K_1 + 2m_2 K_2)$$

$$+ \delta_{i+j,\bar{1}} \delta_{m_1+n_1+1,0} \delta_{m_2+n_2,0}(-1)^j q^{m_2(2n_1+1)} K_1.$$

On the other hand, we have

$$[t_0^i t_1^{2m_1+1} t_2^{m_2}, t_0^j t_1^{2n_1+1} t_2^{n_2}] = \left((-1)^j q^{m_2(2n_1+1)} - (-1)^i q^{(2m_1+1)n_2}\right) t_0^{i+j} t_1^{2m_1+2n_1+2} t_2^{m_2+n_2}$$

$$+ \delta_{i+j,\bar{0}} \delta_{2m_1+2n_1+2,0} \delta_{m_2+n_2,0}(-1)^j q^{m_2(2n_1+1)}((2m_1+1)c_1 + m_2 c_2).$$

Thus

$$\varphi([t_0^i t_1^{2m_1+1} t_2^{m_2}, t_0^j t_1^{2n_1+1} t_2^{n_2}]) = [\varphi(t_0^i t_1^{2m_1+1} t_2^{m_2}), \varphi(t_0^j t_1^{2n_1+1} t_2^{n_2})].$$

4

Similarly, we have

$$[\varphi(t_0^i t_1^{2m_1} t_2^{m_2}), \varphi(t_0^j t_1^{2n_1} t_2^{n_2})]$$

$$= [(-1)^i E_{11}(t_1^{m_1} t_2^{m_2}) + q^{-m_2} E_{22}(t_1^{m_1} t_2^{m_2}), (-1)^j E_{11}(t_1^{n_1} t_2^{n_2}) + q^{-n_2} E_{22}(t_1^{n_1} t_2^{n_2})]$$

$$= (q^{2m_2 n_1} - q^{2n_2 m_1})\left((-1)^{i+j} E_{11}(t_1^{m_1+n_1} t_2^{m_2+n_2}) + q^{-m_2-n_2} E_{22}(t_1^{m_1+n_1} t_2^{m_2+n_2})\right)$$

$$+ \delta_{m_1+n_1,0}\delta_{m_2+n_2,0}\delta_{i+j,\bar{0}} q^{2m_2 n_1}(2m_1 K_1 + 2m_2 K_2),$$

and

$$[t_0^i t_1^{2m_1} t_2^{m_2}, t_0^j t_1^{2n_1} t_2^{n_2}] = (q^{2m_2 n_1} - q^{2m_1 n_2}) t_0^{i+j} t_1^{2m_1+2n_1} t_2^{m_2+n_2}$$

$$+ \delta_{i+j,\bar{0}}\delta_{m_1+n_1,0}\delta_{m_2+n_2,0} q^{2m_2 n_1}(2m_1 c_1 + m_2 c_2).$$

Therefore

$$[\varphi(t_0^i t_1^{2m_1} t_2^{m_2}), \varphi(t_0^j t_1^{2n_1} t_2^{n_2})] = \varphi([t_0^i t_1^{2m_1} t_2^{m_2}, t_0^j t_1^{2n_1} t_2^{n_2}]).$$

Finally, we have

$$[\varphi(t_0^i t_1^{2m_1+1} t_2^{m_2}), \varphi(t_0^j t_1^{2n_1} t_2^{n_2})]$$

$$= [(-1)^i q^{-m_2} E_{12}(t_1^{m_1} t_2^{m_2}) + E_{21}(t_1^{m_1+1} t_2^{m_2}), (-1)^j E_{11}(t_1^{n_1} t_2^{n_2}) + q^{-n_2} E_{22}(t_1^{n_1} t_2^{n_2})]$$

$$= \left((-1)^j q^{2m_2 n_1} - q^{n_2(2m_1+1)}\right)\left((-1)^{i+j} q^{-m_2-n_2} E_{12}(t_1^{m_1+n_1} t_2^{m_2+n_2}) + E_{21}(t_1^{m_1+n_1+1} t_2^{m_2+n_2})\right),$$

and

$$[t_0^i t_1^{2m_1+1} t_2^{m_2}, t_0^j t_1^{2n_1} t_2^{n_2}] = ((-1)^j q^{2m_2 n_1} - q^{n_2(2m_1+1)}) t_0^{i+j} t_1^{2m_1+2n_1+1} t_2^{m_2+n_2}.$$

Thus

$$[\varphi(t_0^i t_1^{2m_1+1} t_2^{m_2}), \varphi(t_0^j t_1^{2n_1} t_2^{n_2})] = \varphi([t_0^i t_1^{2m_1+1} t_2^{m_2}, t_0^j t_1^{2n_1} t_2^{n_2}]).$$

This completes the proof. $\qquad\square$

Remark 2.2 From the proof of above proposition, one can easily see that $gl_2(\mathbf{C}_{q^2}) \cong \widetilde{L}$ and $\overline{\tau} \cong \overline{L}$.

Next we will recall some concepts about the **Z**-graded L-modules. Fix a **Z**-basis

$$\mathbf{m}_1 = (m_{11}, m_{12}), \ \mathbf{m}_2 = (m_{21}, m_{22}) \in \mathbf{Z}^2.$$

If we define the degree of the elements in $\langle t_0^i t^{j\mathbf{m}_1+k\mathbf{m}_2} \in L \mid i \in \mathbf{Z}_2, k \in \mathbf{Z}\rangle$ to be j and the degree of the elements in $\langle c_1, c_2\rangle$ to be zero, then L can be regarded as a **Z**-graded Lie algebra:

$$L_j = \langle t_0^i t^{j\mathbf{m}_1+k\mathbf{m}_2} \in L \mid i \in \mathbf{Z}_2, k \in \mathbf{Z}\rangle \oplus \delta_{j,0}\langle c_1, c_2\rangle.$$

Set

$$L_+ = \bigoplus_{j\in\mathbf{N}} L_j, \quad L_- = \bigoplus_{-j\in\mathbf{N}} L_j.$$

5

Then $L = \oplus_{j \in \mathbf{Z}} L_j$ and L has the following triangular decomposition

$$L = L_- \oplus L_0 \oplus L_+.$$

Definition   For any L-module V, if $V = \oplus_{m \in \mathbf{Z}} V_m$ with

$$L_j \cdot V_m \subset V_{m+j}, \ \forall \ j, m \in \mathbf{Z},$$

then V is called a **Z**-*graded L-module* and V_m is called a *homogeneous subspace of V with degree* $m \in \mathbf{Z}$. The L-module V is called

 (i) a *quasi-finite* **Z**-*graded module* if $\dim V_m < \infty, \ \forall \, m \in \mathbf{Z}$;

 (ii) a *uniformly bounded module* if there exists some $N \in \mathbf{N}$ such that $\dim V_m \leq N, \ \forall \, m \in \mathbf{Z}$;

 (iii) a *highest* (resp. *lowest*) *weight module* if there exists a nonzero homogeneous vector $v \in V_m$ such that V is generated by v and $L_+ \cdot v = 0$ (resp. $L_- \cdot v = 0$);

 (iv) a *generalized highest weight module with highest degree m* (see, e.g., [31]) if there exist a **Z**-basis $B = \{\mathbf{b_1}, \mathbf{b_2}\}$ of $\mathbf{Z}^2$ and a nonzero vector $v \in V_m$ such that V is generated by v and $t_0^i t^{\mathbf{m}} \cdot v = 0, \forall \ \mathbf{m} \in \mathbf{Z}_+ \mathbf{b_1} + \mathbf{Z}_+ \mathbf{b_2}, i \in \mathbf{Z}_2$;

 (v) an *irreducible* **Z**-*graded module* if V does not have any nontrivial **Z**-graded submodule (see, e.g., [29]).

We denote the set of quasi-finite irreducible **Z**-graded L-modules by $\mathcal{O}_{\mathbf{Z}}$. From the definition, one sees that the generalized highest weight modules contain the highest weight modules and the lowest weight modules as their special cases. As the central elements c_1, c_2 of L act on irreducible graded modules V as scalars, we shall use the same symbols to denote these scalars.

Now we study the structure and representations of L_0. Note that by the theory of Verma modules, the irreducible **Z**-graded highest (or lowest) weight L-modules are classified by the characters of L_0.

Lemma 2.3   (1) *If m_{21} is an even integer then L_0 is a Heisenberg Lie algebra.*
(2) *If m_{21} is an odd integer then*

$$L_0 = (\mathcal{A} + \mathcal{B}) \oplus \langle m_{11} c_1 + m_{12} c_2 \rangle,$$

where $\mathcal{A} = \langle t_0^{\bar{0}} t^{2j\mathbf{m_2}}, m_{21} c_1 + m_{22} c_2 \mid j \in \mathbf{Z} \rangle$ is a Heisenberg Lie algebra and

$$\mathcal{B} = \langle t_0^{\bar{1}} t^{j\mathbf{m_2}}, t_0^{\bar{0}} t^{(2j+1)\mathbf{m_2}}, m_{21} c_1 + m_{22} c_2 \mid j \in \mathbf{Z} \rangle,$$

which is isomorphic to the affine Lie algebra $A_1^{(1)}$ and the isomorphism is given by the linear extension of the following map ϕ:

$$t_0^{\bar{1}} t^{2j\mathbf{m_2}} \quad \mapsto \quad -q^{-2j^2 m_{22} m_{21}} \big((E_{11} - E_{22})(x^j) + \frac{1}{2} K \big), \tag{1}$$

6

$$t_0^i t^{(2j+1)\mathbf{m}_2} \quad \mapsto \quad q^{-\frac{1}{2}(2j+1)^2 m_{22} m_{21}} ((-1)^i E_{12}(x^j) + E_{21}(x^{j+1})), \tag{2}$$

$$m_{21} c_1 + m_{22} c_2 \quad \mapsto \quad K. \tag{3}$$

Moreover, we have $[\mathcal{A}, \mathcal{B}] = 0$.

Proof Statement (1) can be easily deduced from the definition of L_0.

(2) To show $\mathcal{B} \cong A_1^{(1)}$, we need to prove that ϕ preserves Lie bracket. Notice that

$$\left[q^{-\frac{1}{2}(2j+1)^2 m_{22} m_{21}} \left((-1)^i E_{12}(x^j) + E_{21}(x^{j+1}) \right), q^{-\frac{1}{2}(2l+1)^2 m_{22} m_{21}} \left((-1)^k E_{12}(x^l) + E_{21}(x^{l+1}) \right) \right]$$
$$= q^{-\frac{1}{2}((2j+1)^2 + (2l+1)^2) m_{22} m_{21}} \Big(((-1)^i - (-1)^k)(E_{11} - E_{22})(x^{j+l+1})$$
$$+ \delta_{j+l+1,0}((-1)^i j + (-1)^k (j+1)) K \Big),$$

and

$$[t_0^i t^{(2j+1)\mathbf{m}_2}, t_0^k t^{(2k+1)\mathbf{m}_2}] = ((-1)^k - (-1)^i) q^{(2j+1)(2k+1) m_{22} m_{21}} t_0^{i+k} t^{(2j+2k+2)\mathbf{m}_2}$$
$$+ \delta_{i+k,\bar{0}} \delta_{j+k+1,0} (-1)^k q^{(2j+1)(2k+1) m_{22} m_{21}} (2j+1)(m_{21} c_1 + m_{22} c_2).$$

One sees that

$$\phi([t_0^i t^{(2j+1)\mathbf{m}_2}, t_0^k t^{(2k+1)\mathbf{m}_2}]) = [\phi(t_0^i t^{(2j+1)\mathbf{m}_2}), \phi(t_0^k t^{(2k+1)\mathbf{m}_2})].$$

Consider

$$[-q^{-2j^2 m_{22} m_{21}} ((E_{11} - E_{22})(x^j) + \tfrac{1}{2} K), q^{-\frac{1}{2}(2l+1)^2 m_{22} m_{21}} ((-1)^k E_{12}(x^l) + E_{21}(x^{l+1}))]$$
$$= -q^{-\frac{1}{2}(4j^2 + (2l+1)^2) m_{22} m_{21}} (2(-1)^k E_{12}(x^{l+j}) - 2 E_{21}(x^{l+j+1}))$$

and

$$[t_0^{\bar{1}} t^{2j\mathbf{m}_2}, t_0^k t^{(2l+1)\mathbf{m}_2}] = 2 q^{2j(2l+1) m_{22} m_{21}} t_0^{k+\bar{1}} t^{(2j+2l+1)\mathbf{m}_2},$$

we have

$$\phi([t_0^{\bar{1}} t^{2j\mathbf{m}_2}, t_0^k t^{(2l+1)\mathbf{m}_2}]) = [\phi(t_0^{\bar{1}} t^{2j\mathbf{m}_2}), \phi(t_0^k t^{(2l+1)\mathbf{m}_2})].$$

Finally, we have

$$[-q^{-2j^2 m_{22} m_{21}} ((E_{11} - E_{22})(x^j) + \tfrac{1}{2} K), -q^{-2l^2 m_{22} m_{21}} ((E_{11} - E_{22})(x^l) + \tfrac{1}{2} K)]$$
$$= 2jq^{-2(j^2 + l^2) m_{22} m_{21}} \delta_{j+l,0} K = 2jq^{4jl m_{22} m_{21}} \delta_{j+l,0} K,$$

and

$$[t_0^{\bar{1}} t^{2j\mathbf{m}_2}, t_0^{\bar{1}} t^{2l\mathbf{m}_2}] = 2jq^{4jl m_{22} m_{21}} \delta_{j+l,0} (m_{21} c_1 + m_{22} c_2).$$

Thus

$$\phi([t_0^{\bar{1}} t^{2j\mathbf{m}_2}, t_0^{\bar{1}} t^{2l\mathbf{m}_2}]) = [\phi(t_0^{\bar{1}} t^{2j\mathbf{m}_2}), \phi(t_0^{\bar{1}} t^{2l\mathbf{m}_2})].$$

7

This proves $\mathcal{B} \cong A_1^{(1)}$. And the proof of the rest results in this lemma is straightforward. $\quad\square$

Since the Lie subalgebra $\mathcal{B}$ of L_0 is isomorphic to the affine Lie algebra $A_1^{(1)}$, we need to collect some results on the finite dimensional irreducible modules of $A_1^{(1)}$ from [30].

Let $\nu > 0$ and $\underline{\mu} = (\mu_1, \mu_2, \cdots, \mu_\nu)$ be a finite sequence of nonzero distinct numbers. Let V_i, $1 \le i \le \nu$ be finite dimensional irreducible sl_2-modules. We define an $A_1^{(1)}$-module $V(\underline{\mu}) = V_1 \otimes V_2 \otimes \cdots \otimes V_\nu$ as follows, for $X \in sl_2, j \in \mathbf{Z}$,

$$X(x^j) \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_\nu) = \sum_{i=1}^{\nu} \mu_i^j v_1 \otimes \cdots \otimes (X \cdot v_i) \otimes \cdots \otimes v_\nu, \quad K \cdot (v_1 \otimes \cdots \otimes v_\nu) = 0.$$

Clearly $V(\underline{\mu})$ is a finite dimensional irreducible $A_1^{(1)}$-module. For any $Q(x) \in \mathbf{C}[x^{\pm 1}]$, we have

$$X(Q(x)) \cdot (V_1 \otimes \cdots \otimes V_\nu) = 0, \ \forall \ X \in sl_2 \iff \prod_{i=1}^{\nu}(x - \mu_1) \mid Q(x).$$

Now by Lemma 2.3(2), if m_{21} is an odd integer then we can define a finite dimensional irreducible L_0-module $V(\underline{\mu}, \psi) = V_1 \otimes \cdots \otimes V_\nu$ as follows

$$t_0^{\bar{0}} t^{2j\mathbf{m}_2} \cdot (v_1 \otimes \cdots \otimes v_\nu) = \psi(t_0^{\bar{0}} t^{2j\mathbf{m}_2}) \cdot (v_1 \otimes \cdots \otimes v_\nu),$$

$$t_0^{\bar{1}} t^{2j\mathbf{m}_2} \cdot (v_1 \otimes \cdots \otimes v_\nu) = -q^{-2j^2 m_{22} m_{21}} \sum_{i=1}^{\nu} \mu_i^j v_1 \otimes \cdots \otimes ((E_{11} - E_{22}) \cdot v_i) \otimes \cdots \otimes v_\nu,$$

$$t_0^{i} t^{(2j+1)\mathbf{m}_2} \cdot (v_1 \otimes \cdots \otimes v_\nu) = q^{-\frac{1}{2}(2j+1)^2 m_{22} m_{21}} \Big((-1)^i \sum_{i=1}^{\nu} \mu_i^j v_1 \otimes \cdots \otimes (E_{12} \cdot v_i) \otimes \cdots \otimes v_\nu$$

$$+ \sum_{i=1}^{\nu} \mu_i^{j+1} v_1 \otimes \cdots \otimes (E_{21} \cdot v_i) \otimes \cdots \otimes v_\nu \Big),$$

$$(m_{21} c_1 + m_{22} c_2) \cdot (v_1 \otimes \cdots \otimes v_\nu) = 0, \quad \forall \ v_1 \otimes \cdots \otimes v_\nu \in V(\underline{\mu}, \psi), j \in \mathbf{Z}, i \in \mathbf{Z}_2,$$

where $\nu > 0$, $\underline{\mu} = (\mu_1, \mu_2, \cdots, \mu_\nu)$ is a finite sequence of nonzero distinct numbers, V_i, $1 \le i \le \nu$ are finite dimensional irreducible sl_2-modules, and ψ is a linear function over $\mathcal{A}$.

Theorem 2.4 ([30, Theorem 2.14]) *Let V be a finite dimensional irreducible $A_1^{(1)}$-module. Then V is isomorphic to $V(\underline{\mu})$ for some finite dimensional irreducible sl_2-modules $V_1, \cdots, V_\nu$ and a finite sequence of nonzero distinct numbers $\underline{\mu} = (\mu_1, \cdots, \mu_\nu)$.*

From the above theorem and Lemma 2.3, we have the following theorem.

Theorem 2.5 *Let m_{21} be an odd integer and V be a finite dimensional irreducible L_0-module. Then V is isomorphic to $V(\underline{\mu}, \psi)$, where $V_1, \cdots, V_\nu$ are some finite dimensional irreducible sl_2-modules, $\underline{\mu} = (\mu_1, \cdots, \mu_\nu)$ is a finite sequence of nonzero distinct numbers, and ψ is a linear function over $\mathcal{A}$.*

8

Remark 2.6 Let m_{21} be an odd integer and $V(\underline{\mu}, \psi)$ be a finite dimensional irreducible L_0-modules defined as above. One can see that for any $k \in \mathbf{Z}_2$,

$$(\sum_{i=1}^{n} b_i q^{\frac{1}{2}(2i+1)^2 m_{22} m_{21}} t_0^k t^{(2i+1)\mathbf{m}_2}) \cdot (V_1 \otimes \cdots \otimes V_\nu) = 0, \quad \text{and}$$

$$(\sum_{i=1}^{n} b_i q^{2i^2 m_{22} m_{21}} t_0^{\bar{1}} t^{2i\mathbf{m}_2}) \cdot (V_1 \otimes \cdots \otimes V_\nu) = 0,$$

if and only if $\prod_{i=1}^{\nu}(x - \mu_1) \mid (\sum_{i=1}^{n} b_i x^i)$.

At the end of this section, we will prove a proposition which reduces the classification of the irreducible $\mathbf{Z}$-graded modules with finite dimensional homogeneous subspaces to that of the generalized highest weight modules and the uniformly bounded modules.

Proposition 2.7 *If V is an irreducible $\mathbf{Z}$-graded L-module, then V is a generalized highest weight module or a uniformly bounded module.*

Proof Let $V = \oplus_{m \in \mathbf{Z}} V_m$. We first prove that if there exists a $\mathbf{Z}$-basis $\{\mathbf{b}_1, \mathbf{b}_2\}$ of $\mathbf{Z}^2$ and a homogeneous vector $v \neq 0$ such that $t_0^i t^{\mathbf{b}_1} \cdot v = t_0^i t^{\mathbf{b}_2} \cdot v = 0$, $\forall i \in \mathbf{Z}/2\mathbf{Z}$, then V is a generalized highest weight modules.

To obtain this, we first introduce the following notation: For any $A \subset \mathbf{Z}^2$, we use t^A to denote the set $\{t^a | a \in A\}$.

Then one can deduce that $t_0^i t^{\mathbf{N}\mathbf{b}_1 + \mathbf{N}\mathbf{b}_2} \cdot v = 0$, $\forall i \in \mathbf{Z}/2\mathbf{Z}$ by the assumption. Thus for the $\mathbf{Z}$-basis $\mathbf{m}_1 = 3\mathbf{b}_1 + \mathbf{b}_2$, $\mathbf{m}_2 = 2\mathbf{b}_1 + \mathbf{b}_2$ of $\mathbf{Z}^2$ we have $t_0^i t^{\mathbf{Z}_+ \mathbf{m}_1 + \mathbf{Z}_+ \mathbf{m}_2} v = 0$, $\forall i \in \mathbf{Z}_2$. Therefore V is a generalized highest weight module by the definition.

With the above statement, we can prove our proposition now. Suppose that V is not a generalized highest weight module. For any $m \in \mathbf{Z}$, considering the following maps

$$\begin{array}{llll}
t_0^{\bar{0}} t^{-m\mathbf{m}_1 + \mathbf{m}_2} : & V_m \mapsto V_0, & t_0^{\bar{1}} t^{-m\mathbf{m}_1 + \mathbf{m}_2} : & V_m \mapsto V_0, \\
t_0^{\bar{0}} t^{(1-m)\mathbf{m}_1 + \mathbf{m}_2} : & V_m \mapsto V_1, & t_0^{\bar{1}} t^{(1-m)\mathbf{m}_1 + \mathbf{m}_2} : & V_m \mapsto V_1,
\end{array}$$

one can easily check that

$$\ker t_0^{\bar{0}} t^{-m\mathbf{m}_1 + \mathbf{m}_2} \cap \ker t_0^{\bar{0}} t^{(1-m)\mathbf{m}_1 + \mathbf{m}_2} \cap \ker t_0^{\bar{1}} t^{-m\mathbf{m}_1 + \mathbf{m}_2} \cap \ker t_0^{\bar{1}} t^{(1-m)\mathbf{m}_1 + \mathbf{m}_2} = \{0\}.$$

Therefore $\dim V_m \leq 2\dim V_0 + 2\dim V_1$. So V is a uniformly bounded module. $\qquad \square$

§3 The highest weight irreducible Z-graded L-modules

For any finite dimensional irreducible L_0-module V, we can define it as a $(L_0 + L_+)$-module by putting $L_+ v = 0$, $\forall v \in V$. Then we obtain an induced L-module,

$$\overline{M}^+(V, \mathbf{m}_1, \mathbf{m}_2) = \mathrm{Ind}_{L_0 + L_+}^{L} V = U(L) \otimes_{U(L_0 + L_+)} V \simeq U(L_-) \otimes V,$$

where $U(L)$ is the universal enveloping algebra of L. If we set V to be the homogeneous subspace of $\overline{M}^+(V, \mathbf{m}_1, \mathbf{m}_2)$ with degree 0, then $\overline{M}^+(V, \mathbf{m}_1, \mathbf{m}_2)$ becomes a $\mathbf{Z}$-graded L-module in a natural way. Obviously, $\overline{M}^+(V, \mathbf{m}_1, \mathbf{m}_2)$ has an unique maximal proper submodule J which trivially intersects with V. So we obtain an irreducible $\mathbf{Z}$-graded highest weight L-module,

$$M^+(V, \mathbf{m}_1, \mathbf{m}_2) = \overline{M}^+(V, \mathbf{m}_1, \mathbf{m}_2)/J.$$

We can write it as

$$M^+(V, \mathbf{m}_1, \mathbf{m}_2) = \bigoplus_{i \in \mathbf{Z}_+} V_{-i},$$

where V_{-i} is the homogeneous subspaces of degree $-i$. Since L_- is generated by L_{-1}, and L_+ is generated by L_1, by the construction of $M^+(V, \mathbf{m}_1, \mathbf{m}_2)$, we see that

$$L_{-1}V_{-i} = V_{-i-1}, \quad \forall\, i \in \mathbf{Z}_+, \tag{3.1}$$

and for a homogeneous vector v,

$$L_1 \cdot v = 0 \implies v = 0. \tag{3.2}$$

Similarly, we can define an irreducible lowest weight $\mathbf{Z}$-graded L-module $M^-(V, \mathbf{m}_1, \mathbf{m}_2)$ from any finite dimensional irreducible L_0-module V.

If $m_{21} \in \mathbf{Z}$ is even then L_0 is a Heisenberg Lie algebra by Lemma 2.3. Thus, by a well-known result about the representations of the Heisenberg Lie algebra, we see that the finite dimensional irreducible L_0-module V must be a one dimensional module $\mathbf{C}v_0$, and there is a linear function ψ over L_0 such that

$$t_0^i t^{j\mathbf{m_2}} \cdot v_0 = \psi(t_0^i t^{j\mathbf{m_2}}) \cdot v_0, \;\; \psi(m_{21}c_1 + m_{22}c_2) = 0, \; \forall\, i \in \mathbf{Z}_2, j \in \mathbf{Z}.$$

In this case, we denote the corresponding highest weight, resp., lowest weight, irreducible $\mathbf{Z}$-graded L-module by

$$M^+(\psi, \mathbf{m}_1, \mathbf{m}_2), \quad \text{resp.,} \quad M^-(\psi, \mathbf{m}_1, \mathbf{m}_2).$$

If m_{21} is an odd integer then V must be isomorphic to $V(\underline{\mu}, \psi)$ by Theorem 2.5. We denote the corresponding highest weight, resp. lowest weight, irreducible $\mathbf{Z}$-graded L-module by

$$M^+(\underline{\mu}, \psi, \mathbf{m}_1, \mathbf{m}_2), \quad \text{resp.,} \quad M^-(\underline{\mu}, \psi, \mathbf{m}_1, \mathbf{m}_2).$$

The irreducible $\mathbf{Z}$-graded L-modules $M^+(\psi, \mathbf{m}_1, \mathbf{m}_2)$ and $M^+(\underline{\mu}, \psi, \mathbf{m}_1, \mathbf{m}_2)$ are in general not quasi-finite modules. Thus in the rest of this section we shall determine which of $\underline{\mu}$ and ψ can correspond to quasi-finite modules.

For the later use, we obtain the following equations from the definition of L, where, $\alpha = m_{11}m_{22} - m_{12}m_{21} \in \{\pm 1\}$,

$$[t_0^j t^{\mathbf{m}_1+k\mathbf{m}_2}, t_0^r t^{-\mathbf{m}_1+s\mathbf{m}_2} t^{i\mathbf{m}_2}]$$

$$= q^{i(-m_{12}+sm_{22})m_{21}}[t_0^j t^{\mathbf{m}_1+k\mathbf{m}_2}, t_0^r t^{-\mathbf{m}_1+(s+i)\mathbf{m}_2}]$$

$$= q^{-m_{11}m_{12}-km_{11}m_{22}+sm_{12}m_{21}+ksm_{21}m_{22}}(-1)^{r(m_{11}+km_{21})} \times$$

$$\times \left((1-(-1)^{(j+r)m_{11}+(kr+js+ji)m_{21}} q^{(k+s+i)\alpha}) t_0^{j+r} t^{(k+s)\mathbf{m}_2} t^{i\mathbf{m}_2} \right.$$

$$\left. +\delta_{k+s+i,0}\delta_{j+r,\bar{0}} q^{-(k+s)^2 m_{21}m_{22}}((m_{11}+km_{21})c_1 + (m_{12}+km_{22})c_2) \right), \qquad (3)$$

$$[t_0^s t^{k\mathbf{m}_2} t^{i\mathbf{m}_2}, t_0^r t^{-\mathbf{m}_1+j\mathbf{m}_2}]$$

$$= q^{kim_{22}m_{21}}[t_0^s t^{(k+i)\mathbf{m}_2}, t_0^r t^{-\mathbf{m}_1+j\mathbf{m}_2}]$$

$$= q^{km_{22}(-m_{11}+jm_{21})}(-1)^{(rk+ri)m_{21}}(q^{-i\alpha} - (-1)^{sm_{11}+(rk+ri+sj)m_{21}} q^{k\alpha}) \times$$

$$\times t_0^{r+s} t^{-\mathbf{m}_1+(k+j)\mathbf{m}_2} t^{i\mathbf{m}_2}. \qquad (4)$$

In the rest of this section, if $P(t^{\mathbf{m}_2}) = \sum_{i=0}^n a_i t^{i\mathbf{m}_2} \in \mathbf{C}[t^{\mathbf{m}_2}]$ then we will denote $\sum_{i=0}^n a_i b^i t^{i\mathbf{m}_2}$ by $P(bt^{\mathbf{m}_2})$ for any $b \in \mathbf{C}$.

Lemma 3.1 *Let m_{21} be an even integer. Then $M^{\pm}(\psi, \mathbf{m}_1, \mathbf{m}_2) \in \mathcal{O}_{\mathbf{Z}}$ if and only if there exists a polynomial $P(t^{\mathbf{m}_2}) = \sum_{i=0}^n a_i t^{i\mathbf{m}_2} \in \mathbf{C}[t^{\mathbf{m}_2}]$ with $a_0 a_n \neq 0$ such that for $k \in \mathbf{Z}, j \in \mathbf{Z}_2$,*

$$\psi\left(t_0^j t^{k\mathbf{m}_2} P(t^{\mathbf{m}_2}) - (-1)^j q^{k\alpha} t_0^j t^{k\mathbf{m}_2} P(q^{\alpha} t^{\mathbf{m}_2}) + \delta_{j,\bar{0}} a_{-k} q^{-k^2 m_{21}m_{22}} \beta \right) = 0, \qquad (3.5)$$

where $a_k = 0$ if $k \notin \{0, 1, \cdots, n\}$, and $\alpha = m_{11}m_{22} - m_{12}m_{21} \in \{\pm 1\}$, $\beta = m_{11}c_1 + m_{12}c_2$.

Proof Since m_{21} is an even integer and $m_{11}m_{22} - m_{12}m_{21} \in \{\pm 1\}$, we see m_{11} is an odd integer.

"$\Longrightarrow$". Since $\dim V_{-1} < \infty$, there exist an integer s and a polynomial $P(t^{\mathbf{m}_2}) = \sum_{i=0}^n a_i t^{i\mathbf{m}_2} \in \mathbf{C}[t^{\mathbf{m}_2}]$ with $a_0 a_n \neq 0$ such that

$$t_0^{\bar{0}} t^{-\mathbf{m}_1+s\mathbf{m}_2} P(t^{\mathbf{m}_2}) \cdot v_0 = 0.$$

Applying $t_0^j t^{\mathbf{m}_1+k\mathbf{m}_2}$ for any $k \in \mathbf{Z}, j \in \mathbf{Z}_2$ to the above equation, we have that

$$0 = t_0^j t^{\mathbf{m}_1+k\mathbf{m}_2} \cdot t_0^{\bar{0}} t^{-\mathbf{m}_1+s\mathbf{m}_2} P(t^{\mathbf{m}_2}) \cdot v_0 = \sum_{i=0}^n [t_0^j t^{\mathbf{m}_1+k\mathbf{m}_2}, a_i t_0^{\bar{0}} t^{-\mathbf{m}_1+s\mathbf{m}_2} t^{i\mathbf{m}_2}] \cdot v_0.$$

11

Thus, by (3.3), we have

$$
0 = \psi\left(\sum_{i=0}^{n} a_i\left((1 - (-1)^j q^{(k+s+i)\alpha})t_0^j t^{(k+s)\mathbf{m}_2} t^{i\mathbf{m}_2} + \delta_{k+s+i,0}\delta_{j,\bar{0}}q^{-(k+s)^2 m_{21}m_{22}}\beta\right)\right)
$$

$$
= \psi\left(t_0^j t^{(k+s)\mathbf{m}_2} P(t^{\mathbf{m}_2}) - (-1)^j q^{(k+s)\alpha} t_0^j t^{(k+s)\mathbf{m}_2} P(q^\alpha t^{\mathbf{m}_2}) + a_{-k-s}\delta_{j,\bar{0}}q^{-(k+s)^2 m_{21}m_{22}}\beta\right).
$$

Therefore this direction follows.

"$\Longleftarrow$". By induction on s we first show the following claim.

Claim. *For any $s \in \mathbf{Z}_+$, there exists polynomial $P_s(t^{\mathbf{m}_2}) = \sum_{i\in\mathbf{Z}} a_{s,i} t^{i\mathbf{m}_2} \in \mathbf{C}[t^{\mathbf{m}_2}]$ such that*

$$
\left(t_0^r t^{k\mathbf{m}_2} P_s(t^{\mathbf{m}_2}) - (-1)^r q^{k\alpha} t_0^r t^{k\mathbf{m}_2} P_s(q^\alpha t^{\mathbf{m}_2}) + \delta_{r,\bar{0}} a_{s,-k} q^{-k^2 m_{21}m_{22}}\beta\right) \cdot V_{-s} = 0,
$$

$$
t_0^r t^{-\mathbf{m}_1 + k\mathbf{m}_2} P_s(t^{\mathbf{m}_2}) \cdot V_{-s} = 0, \quad \forall\, r \in \mathbf{Z}_2, k \in \mathbf{Z}.
$$

For $s = 0$, the first equation holds with $P_0(t^{\mathbf{m}_2}) = P(t^{\mathbf{m}_2})$ (with P being as in the necessity), and by (3.2), the second equation can be deduced by a calculation similar to the proof of the necessity. Suppose the claim holds for s. Let us consider the claim for $s + 1$.

Note that the equations in the claim are equivalent to

$$
\left(t_0^r Q(t^{\mathbf{m}_2}) - (-1)^r t_0^r Q(q^\alpha t^{\mathbf{m}_2}) + \delta_{r,\bar{0}} a_Q \beta\right) \cdot V_{-s} = 0,
$$

$$
t_0^r t^{-\mathbf{m}_1 + k\mathbf{m}_2} Q(t^{\mathbf{m}_2}) \cdot V_{-s} = 0, \quad \forall\, r \in \mathbf{Z}_2, k \in \mathbf{Z}, \tag{6}
$$

for any $Q(t^{\mathbf{m}_2}) \in \mathbf{C}[t^{\pm\mathbf{m}_2}]$ with $P_s(t^{\mathbf{m}_2}) \mid Q(t^{\mathbf{m}_2})$, where a_Q is the constant term of $Q(t^{\mathbf{m}_2})$.

Let $P_{s+1}(t^{\mathbf{m}_2}) = P_s(q^\alpha t^{\mathbf{m}_2}) P_s(t^{\mathbf{m}_2}) P_s(q^{-\alpha} t^{\mathbf{m}_2})$, then

$$
P_s(t^{\mathbf{m}_2}) \mid P_{s+1}(t^{\mathbf{m}_2}), \quad P_s(t^{\mathbf{m}_2}) \mid P_{s+1}(q^\alpha t^{\mathbf{m}_2}) \quad \text{and} \quad P_s(t^{\mathbf{m}_2}) \mid P_{s+1}(q^{-\alpha} t^{\mathbf{m}_2}).
$$

For any $p, r \in \mathbf{Z}_2$, $j, k \in \mathbf{Z}$, by induction and (3.4), we have

$$
\left(t_0^r t^{k\mathbf{m}_2} P_{s+1}(t^{\mathbf{m}_2}) - (-1)^r q^{k\alpha} t_0^r t^{k\mathbf{m}_2} P_{s+1}(q^\alpha t^{\mathbf{m}_2}) + \delta_{r,\bar{0}} a_{s+1,-k} q^{-k^2 m_{21}m_{22}}\beta\right) \cdot t_0^p t^{-\mathbf{m}_1 + j\mathbf{m}_2} \cdot V_{-s}
$$

$$
= \left[t_0^r t^{k\mathbf{m}_2} P_{s+1}(t^{\mathbf{m}_2}) - (-1)^r q^{k\alpha} t_0^r t^{k\mathbf{m}_2} P_{s+1}(q^\alpha t^{\mathbf{m}_2}) + \delta_{r,\bar{0}} a_{s+1,-k} q^{-k^2 m_{21}m_{22}}\beta, t_0^p t^{-\mathbf{m}_1 + j\mathbf{m}_2}\right] \cdot V_{-s}
$$

$$
= q^{-km_{22}m_{11} + kjm_{22}m_{21}} \left(t_0^{r+p} t^{-\mathbf{m}_1 + (k+j)\mathbf{m}_2}\left(P_{s+1}(q^{-\alpha} t^{\mathbf{m}_2}) - 2(-1)^r q^{k\alpha} P_{s+1}(t^{\mathbf{m}_2})\right.\right.
$$

$$
\left.\left. + q^{2k\alpha} P_{s+1}(q^\alpha t^{\mathbf{m}_2})\right)\right) \cdot V_{-s}
$$

$$
= 0.
$$

Thus, by (3.1) and (3.2), we obtain that

$$
\left(t_0^r t^{k\mathbf{m}_2} P_{s+1}(t^{\mathbf{m}_2}) - (-1)^r q^{k\alpha} t_0^r t^{k\mathbf{m}_2} P_{s+1}(q^\alpha t^{\mathbf{m}_2}) + \delta_{r,\bar{0}} a_{s+1,-k} q^{-k^2 m_{21}m_{22}}\beta\right) \cdot V_{-s-1} = 0. \tag{3.7}
$$

This proves the first equation in the claim for $i = s + 1$.

Using (3.3), (3.6) and induction, we deduce that for any $l, k \in \mathbf{Z}$, $n, r \in \mathbf{Z}_2$,

$$t_0^n t^{\mathbf{m}_1 + l\mathbf{m}_2} \cdot t_0^r t^{-\mathbf{m}_1 + k\mathbf{m}_2} P_{s+1}(t^{\mathbf{m}_2}) \cdot V_{-s-1}$$

$$= [t_0^n t^{\mathbf{m}_1 + l\mathbf{m}_2}, t_0^r t^{-\mathbf{m}_1 + k\mathbf{m}_2} P_{s+1}(t^{\mathbf{m}_2})] \cdot V_{-s-1} + t_0^r t^{-\mathbf{m}_1 + k\mathbf{m}_2} P_{s+1}(t^{\mathbf{m}_2}) \cdot t_0^n t^{\mathbf{m}_1 + l\mathbf{m}_2} \cdot V_{-s-1}$$

$$= (-1)^r q^{-m_{11}m_{12} + km_{12}m_{21} - lm_{11}m_{22} + lkm_{21}m_{22}} \left(t_0^{n+r} t^{(l+k)\mathbf{m}_2} P_{s+1}(t^{\mathbf{m}_2}) \right.$$

$$\left. - (-1)^{n+r} q^{(k+l)\alpha} t_0^{n+r} t^{(l+k)\mathbf{m}_2} P_{s+1}(q^{\alpha} t^{\mathbf{m}_2}) + a_{s+1,-l-k} \delta_{r+n,\bar{0}} q^{-(l+k)^2 m_{21}m_{22}} \beta \right) \cdot V_{-s-1}$$

$$= 0,$$

since $t_0^n t^{\mathbf{m}_1 + l\mathbf{m}_2} \cdot V_{-s-1} \in V_{-s}$. Hence by (3.2),

$$t_0^r t^{-\mathbf{m}_1 + k\mathbf{m}_2} P_{s+1}(t^{\mathbf{m}_2}) \cdot V_{-s-1} = 0 \text{ for all } r \in \mathbf{Z}_2, \ k \in \mathbf{Z},$$

which implies the second equation in the claim for $i = s + 1$. Therefore the claim follows by induction.

From the second equation of the claim and (3.1), we see that

$$\dim V_{-s-1} \leq 2\deg(P_{s+1}(t^{\mathbf{m}_2})) \cdot \dim V_s, \quad \forall \, s \in \mathbf{Z}_+,$$

where $\deg(P_{s+1}(t^{\mathbf{m}_2}))$ denotes the degree of polynomial $P_{s+1}(t^{\mathbf{m}_2})$. Hence $M^+(\psi, \mathbf{m}_1, \mathbf{m}_2) \in \mathcal{O}_{\mathbf{Z}}$.

Similarly we can prove the statement for $M^-(\psi, \mathbf{m}_1, \mathbf{m}_2)$. $\hfill\square$

Theorem 3.2 *Let m_{21} be an even integer. Then $M^{\pm}(\psi, \mathbf{m}_1, \mathbf{m}_2) \in \mathcal{O}_{\mathbf{Z}}$ if and only if there exist $b_{10}^{(j)}, b_{11}^{(j)}, \cdots, b_{1s_1}^{(j)}, b_{20}^{(j)}, b_{21}^{(j)}, \cdots, b_{2s_2}^{(j)}, \cdots, b_{r0}^{(j)}, b_{r1}^{(j)}, \cdots, b_{rs_r}^{(j)} \in \mathbf{C}$ for $j \in \mathbf{Z}_2$, and $\alpha_1, \cdots, \alpha_r \in \mathbf{C}^*$ such that for any $i \in \mathbf{Z}^*$, $j \in \mathbf{Z}_2$,*

$$\psi(t_0^j t^{i\mathbf{m}_2}) = \frac{(b_{10}^{(j)} + b_{11}^{(j)} i + \cdots + b_{1s_1}^{(j)} i^{s_1})\alpha_1^i + \cdots + (b_{r0}^{(j)} + b_{r1}^{(j)} i + \cdots, b_{rs_r}^{(j)} i^{s_r})\alpha_r^i}{(1 - (-1)^j q^{i\alpha}) q^{\frac{1}{2} i^2 m_{21} m_{22}}},$$

$$\psi(\beta) = b_{10}^{(0)} + b_{20}^{(0)} + \cdots + b_{r0}^{(0)},$$

$$\psi(t_0^{\bar{1}} t^{\mathbf{0}}) = \tfrac{1}{2}(b_{10}^{(1)} + + b_{20}^{(1)} + \cdots + b_{r0}^{(1)}), \quad and \quad \psi(m_{21}c_1 + m_{22}c_2) = 0,$$

where $\alpha = m_{11}m_{22} - m_{21}m_{12} \in \{\pm 1\}$ and $\beta = m_{11}c_1 + m_{12}c_2$.

Proof "$\Longrightarrow$". Let $f_{j,i} = \psi((1 - (-1)^j q^{i\alpha}) q^{\frac{1}{2} i^2 m_{21}m_{22}} t_0^j t^{i\mathbf{m}_2})$ for $j \in \mathbf{Z}_2$, $i \in \mathbf{Z}^*$ and $f_{0,0} = \psi(\beta)$, $f_{1,0} = \psi(2t_0^1 t^{\mathbf{0}})$. By Lemma 3.1 there exist complex numbers $a_0, a_1, \cdots, a_n$ with $a_0 a_n \neq 0$ such that

$$\sum_{i=0}^{n} a_i q^{-\frac{1}{2} i^2 m_{21} m_{22}} f_{j,k+i} = 0, \quad \forall \, k \in \mathbf{Z}, j \in \mathbf{Z}_2. \tag{3.8}$$

13

Denote $b_i = a_i q^{-\frac{1}{2}i^2 m_{21} m_{22}}$. Then the above equation becomes

$$\sum_{i=0}^{n} b_i f_{j,k+i} = 0, \ \ \forall \, k \in \mathbf{Z}, j \in \mathbf{Z}_2. \tag{3.9}$$

Suppose $\alpha_1, \cdots, \alpha_r$ are all distinct roots of the equation $\sum_{i=0}^{n} b_i x^i = 0$ with multiplicity $s_1 + 1, \cdots, s_r + 1$ respectively. By a well-known combinatorial formula, we see that there exist $b_{10}^{(j)}, b_{11}^{(j)}, \cdots, b_{1s_1}^{(j)}, \cdots, b_{r0}^{(j)}, b_{r1}^{(j)}, \cdots, b_{rs_r}^{(j)} \in \mathbf{C}$ for $j \in \mathbf{Z}_2$ such that

$$f_{j,i} = (b_{10}^{(j)} + b_{11}^{(j)} i + \cdots + b_{1s_1}^{(j)} i^{s_1}) \alpha_1^i + \cdots + (b_{r0}^{(j)} + b_{r1}^{(j)} i + \cdots, b_{rs_r}^{(j)} i^{s_r}) \alpha_r^i, \ \ \forall \, i \in \mathbf{Z}.$$

Therefore, for any $i \in \mathbf{Z}^*$, $j \in \mathbf{Z}_2$,

$$\psi(t_0^j t^{i\mathbf{m}_2}) = \frac{(b_{10}^{(j)} + b_{11}^{(j)} i + \cdots + b_{1s_1}^{(j)} i^{s_1}) \alpha_1^i + \cdots + (b_{r0}^{(j)} + b_{r1}^{(j)} i + \cdots, b_{rs_r}^{(j)} i^{s_r}) \alpha_r^i}{(1 - (-1)^j q^{i\alpha}) q^{\frac{1}{2}i^2 m_{21} m_{22}}},$$

$$\psi(\beta) = f_{0,0} = b_{10}^{(0)} + b_{20}^{(0)} + \cdots + b_{r0}^{(0)}, \quad \text{and}$$

$$\psi(t_0^{\bar{1}} t^{\mathbf{0}}) = f_{1,0} = \tfrac{1}{2}(b_{10}^{(1)} + + b_{20}^{(1)} + \cdots + b_{r0}^{(1)}).$$

Thus we obtain the expression as required. This direction follows.

"$\Longleftarrow$". Set

$$Q(x) = \prod_{i=1}^{r} (x - \alpha_i)^{s_i + 1} = \sum_{i=1}^{n} b_i x^i \in \mathbf{C}[x], \ \ f_{j,i} = (1 - (-1)^j q^{i\alpha}) q^{\frac{1}{2}i^2 m_{21} m_{22}} \psi(t_0^j t^{i\mathbf{m}_2}),$$

for $j \in \mathbf{Z}_2$, $i \in \mathbf{Z}^*$, and set

$$f_{0,0} = \psi(\beta), \ \ f_{1,0} = 2\psi(t_0^1 t^{\mathbf{0}}).$$

Then one can verify that (3.9) holds. Let $a_i = q^{\frac{1}{2}i^2 m_{21} m_{22}} b_i$. One deduces that (3.8) holds. Thus (3.5) holds for $P(t^{\mathbf{m}_2}) = \sum_{i=0}^{n} a_i t^{i\mathbf{m}_2}$. Therefore this direction follows by using Lemma 3.1. $\quad \Box$

Lemma 3.3 *If m_{21} is an odd integer, then $M^+(\mu, \psi, \mathbf{m}_1, \mathbf{m}_2) \in \mathcal{O}_{\mathbf{Z}}$ if and only if there exists a polynomial $P(t^{\mathbf{m}_2}) = \sum_{i=0}^{n} a_i t^{2i\mathbf{m}_2} \in \mathbf{C}[t^{\mathbf{m}_2}]$ with $a_0 a_n \neq 0$ such that for any $k \in \mathbf{Z}$ and $v \in V_0$,*

$$\left(t_0^{\bar{0}} t^{2k\mathbf{m}_2} P(t^{\mathbf{m}_2}) - q^{2k\alpha} t_0^{\bar{0}} t^{2k\mathbf{m}_2} P(q^\alpha t^{\mathbf{m}_2}) + a_{-k} q^{-4k^2 m_{21} m_{22}} \beta \right) \cdot v = 0, \tag{10}$$

$$t_0^{\bar{0}} t^{(2k+1)\mathbf{m}_2} P(t^{\mathbf{m}_2}) \cdot v = t_0^{\bar{0}} t^{(2k+1)\mathbf{m}_2} P(q^\alpha t^{\mathbf{m}_2}) \cdot v = 0, \tag{11}$$

$$t_0^{\bar{1}} t^{k\mathbf{m}_2} P(t^{\mathbf{m}_2}) \cdot v = t_0^{\bar{1}} t^{k\mathbf{m}_2} P(q^\alpha t^{\mathbf{m}_2}) \cdot v = 0, \tag{12}$$

where $a_k = 0$ if $k \notin \{0, 1, \cdots, n\}$, and $\alpha = m_{11} m_{22} - m_{12} m_{21}$, $\beta = m_{11} c_1 + m_{12} c_2$.

14

Proof "$\Longrightarrow$". Since V_0 is a finite dimensional irreducible L_0-module, we have $V_0 \cong V(\underline{\mu}, \psi)$ as L_0-modules by Theorem 2.5. Since $\mathcal{H} = \langle t_0^{\bar{1}} t^{2k\mathbf{m}_2} \mid k \in \mathbf{Z} \rangle$ is an Abelian Lie subalgebra of L_0, we can choose a common eigenvector $v_0 \in V_0$ of $\mathcal{H}$. First we prove the following claim.

Claim 1 There is a polynomial $P_e(t^{\mathbf{m}_2}) = \sum_{i=0}^{n} a_i t^{2i\mathbf{m}_2}$ with $a_n a_0 \neq 0$ such that

$$\left(t_0^{\bar{0}} t^{2k\mathbf{m}_2} Q(t^{\mathbf{m}_2}) - q^{2k\alpha} t_0^{\bar{0}} t^{2k\mathbf{m}_2} Q(q^{\alpha} t^{\mathbf{m}_2}) + a_Q \beta \right) \cdot v_0 = 0,$$

$$\left(t_0^{\bar{1}} t^{2k\mathbf{m}_2} Q(t^{\mathbf{m}_2}) - (-1)^{m_{11}} q^{2k\alpha} t_0^{\bar{1}} t^{2k\mathbf{m}_2} Q(q^{\alpha} t^{\mathbf{m}_2}) \right) \cdot v_0 = 0,$$

$$\left(t_0^{\bar{0}} t^{(2k+1)\mathbf{m}_2} Q(t^{\mathbf{m}_2}) - q^{(2k+1)\alpha} t_0^{\bar{0}} t^{(2k+1)\mathbf{m}_2} Q(q^{\alpha} t^{\mathbf{m}_2}) \right) \cdot v_0 = 0,$$

$$\left(t_0^{\bar{1}} t^{(2k+1)\mathbf{m}_2} Q(t^{\mathbf{m}_2}) - (-1)^{m_{11}} q^{(2k+1)\alpha} t_0^{\bar{0}} t^{(2k+1)\mathbf{m}_2} Q(q^{\alpha} t^{\mathbf{m}_2}) \right) \cdot v_0 = 0, \tag{13}$$

for all $k \in \mathbf{Z}$ and $Q(t^{\mathbf{m}_2}) \in \mathbf{C}[t^{\pm 2\mathbf{m}_2}]$ with $P_e(t^{\mathbf{m}_2}) \mid Q(t^{\mathbf{m}_2})$, where a_Q is the constant term of $t^{2k\mathbf{m}_2} Q(t^{\mathbf{m}_2})$.

To prove the claim, since $\dim V_{-1} < \infty$, there exist an integer s and a polynomial $P_e(t^{\mathbf{m}_2}) = \sum_{i=0}^{n} a_i t^{2i\mathbf{m}_2} \in \mathbf{C}[t^{\mathbf{m}_2}]$ with $a_0 a_n \neq 0$ such that

$$t_0^{\bar{0}} t^{-\mathbf{m}_1 + 2s\mathbf{m}_2} P_e(t^{\mathbf{m}_2}) \cdot v_0 = 0. \tag{3.14}$$

Applying $t_0^{\bar{0}} t^{\mathbf{m}_1 + 2k\mathbf{m}_2}$ for any $k \in \mathbf{Z}$ to the above equation, we have

$$0 = t_0^{\bar{0}} t^{\mathbf{m}_1 + 2k\mathbf{m}_2} \cdot t_0^{\bar{0}} t^{-\mathbf{m}_1 + 2s\mathbf{m}_2} P_e(t^{\mathbf{m}_2}) \cdot v_0$$

$$= \sum_{i=0}^{n} a_i [t_0^{\bar{0}} t^{\mathbf{m}_1 + 2k\mathbf{m}_2}, q^{2im_{21}(-m_{12} + 2sm_{22})} t_0^{\bar{0}} t^{-\mathbf{m}_1 + 2(s+i)\mathbf{m}_2}] \cdot v_0$$

$$= q^{-m_{11}m_{12} - 2km_{22}m_{11} + 2sm_{12}m_{21} + 4ksm_{21}m_{22}} \times$$

$$\times \left(t_0^{\bar{0}} t^{2(k+s)\mathbf{m}_2} P_e(t^{\mathbf{m}_2}) - q^{2(s+k)\alpha} t_0^{\bar{0}} t^{2(k+s)\mathbf{m}_2} P_e(q^{\alpha} t^{\mathbf{m}_2}) + a_{-k-s} q^{-4(k+s)^2 m_{21} m_{22}} \beta \right) \cdot v_0. \tag{15}$$

Now applying $t_0^{\bar{1}} t^{\mathbf{m}_1 + 2k\mathbf{m}_2}$ for any $k \in \mathbf{Z}$ to (3.14), we have

$$0 = t_0^{\bar{1}} t^{\mathbf{m}_1 + 2k\mathbf{m}_2} \cdot t_0^{\bar{0}} t^{-\mathbf{m}_1 + 2s\mathbf{m}_2} P_e(t^{\mathbf{m}_2}) \cdot v_0$$

$$= \sum_{i=0}^{n} a_i [t_0^{\bar{1}} t^{\mathbf{m}_1 + 2k\mathbf{m}_2}, q^{2im_{21}(-m_{12} + 2sm_{22})} t_0^{\bar{0}} t^{-\mathbf{m}_1 + 2(s+i)\mathbf{m}_2}] \cdot v_0$$

$$= q^{-m_{11}m_{12} - 2km_{22}m_{11} + 2sm_{12}m_{21} + 4ksm_{21}m_{22}} \times$$

$$\times \left(t_0^{\bar{1}} t^{2(k+s)\mathbf{m}_2} P_e(t^{\mathbf{m}_2}) - (-1)^{m_{11}} q^{2(s+k)\alpha} t_0^{\bar{1}} t^{2(k+s)\mathbf{m}_2} P_e(q^{\alpha} t^{\mathbf{m}_2}) \right) \cdot v_0. \tag{16}$$

By applying $t_0^{\bar{0}} t^{\mathbf{m}_1+(2k+1)\mathbf{m}_2}$ and $t_0^{\bar{1}} t^{\mathbf{m}_1+(2k+1)\mathbf{m}_2}$ to (3.14) respectively, one gets that

$$0 = t_0^{\bar{0}} t^{\mathbf{m}_1+(2k+1)\mathbf{m}_2} \cdot t_0^{\bar{0}} t^{-\mathbf{m}_1+2s\mathbf{m}_2} P_e(t^{\mathbf{m}_2}) \cdot v_0$$

$$= q^{-m_{11}m_{12}-(2k+1)m_{11}m_{22}+2sm_{12}m_{21}+2s(2k+1)m_{21}m_{22}} \times$$

$$\times \left(t_0^{\bar{0}} t^{(2k+2s+1)\mathbf{m}_2} P_e(t^{\mathbf{m}_2}) - q^{(2k+2s+1)\alpha} t_0^{\bar{0}} t^{(2k+2s+1)\mathbf{m}_2} P_e(q^\alpha t^{\mathbf{m}_2}) \right) \cdot v_0, \tag{17}$$

$$0 = t_0^{\bar{1}} t^{\mathbf{m}_1+(2k+1)\mathbf{m}_2} \cdot \left(t_0^{\bar{0}} t^{-\mathbf{m}_1+2s\mathbf{m}_2} P_e(t^{\mathbf{m}_2}) \right) \cdot v_0$$

$$= q^{-m_{11}m_{12}-(2k+1)m_{11}m_{22}+2sm_{12}m_{21}+2s(2k+1)m_{21}m_{22}} \times$$

$$\times \left(t_0^{\bar{1}} t^{(2k+2s+1)\mathbf{m}_2} P_e(t^{\mathbf{m}_2}) - (-1)^{m_{11}} q^{(2k+2s+1)\alpha} t_0^{\bar{0}} t^{(2k+2s+1)\mathbf{m}_2} P_e(q^\alpha t^{\mathbf{m}_2}) \right) \cdot v_0. \tag{18}$$

So we have

$$\left(t_0^{\bar{0}} t^{2k\mathbf{m}_2} P_e(t^{\mathbf{m}_2}) - q^{2k\alpha} t_0^{\bar{0}} t^{2k\mathbf{m}_2} P_e(q^\alpha t^{\mathbf{m}_2}) + a_{-k} q^{-4k^2 m_{21}m_{22}} \beta \right) \cdot v_0 = 0,$$

$$\left(t_0^{\bar{1}} t^{2k\mathbf{m}_2} P_e(t^{\mathbf{m}_2}) - (-1)^{m_{11}} q^{2k\alpha} t_0^{\bar{1}} t^{2k\mathbf{m}_2} P_e(q^\alpha t^{\mathbf{m}_2}) \right) \cdot v_0 = 0,$$

$$\left(t_0^{\bar{0}} t^{(2k+1)\mathbf{m}_2} P_e(t^{\mathbf{m}_2}) - q^{(2k+1)\alpha} t_0^{\bar{0}} t^{(2k+1)\mathbf{m}_2} P_e(q^\alpha t^{\mathbf{m}_2}) \right) \cdot v_0 = 0,$$

$$\left(t_0^{\bar{1}} t^{(2k+1)\mathbf{m}_2} P_e(t^{\mathbf{m}_2}) - (-1)^{m_{11}} q^{(2k+1)\alpha} t_0^{\bar{0}} t^{(2k+1)\mathbf{m}_2} P_e(q^\alpha t^{\mathbf{m}_2}) \right) \cdot v_0 = 0,$$

for all $k \in \mathbf{Z}$, which deduces the claim as required.

On the other hand, we can choose an integer s and a polynomial $P_o(t^{\mathbf{m}_2}) = \sum_{i=0}^n a_i t^{2i\mathbf{m}_2} \in \mathbf{C}[t^{\mathbf{m}_2}]$ with $a_0 a_n \neq 0$ such that

$$t_0^{\bar{0}} t^{-\mathbf{m}_1+(2s+1)\mathbf{m}_2} P_o(t^{\mathbf{m}_2}) \cdot v_0 = 0,$$

since $\dim V_{-1} < \infty$. Thus by a calculation similar to the proof of Claim 1, we can deduce the following claim.

Claim 2 There is a polynomial $P_o(t^{\mathbf{m}_2}) = \sum_{i=0}^n a_i t^{2i\mathbf{m}_2}$ with $a_n a_0 \neq 0$ such that

$$\left(t_0^{\bar{0}} t^{2k\mathbf{m}_2} Q(t^{\mathbf{m}_2}) - q^{2k\alpha} t_0^{\bar{0}} t^{2k\mathbf{m}_2} Q(q^\alpha t^{\mathbf{m}_2}) + a_Q \beta \right) \cdot v_0 = 0,$$

$$\left(t_0^{\bar{1}} t^{2k\mathbf{m}_2} Q(t^{\mathbf{m}_2}) - (-1)^{m_{11}+1} q^{2k\alpha} t_0^{\bar{1}} t^{2k\mathbf{m}_2} Q(q^\alpha t^{\mathbf{m}_2}) \right) \cdot v_0 = 0,$$

$$\left(t_0^{\bar{0}} t^{(2k+1)\mathbf{m}_2} Q(t^{\mathbf{m}_2}) - q^{(2k+1)\alpha} t_0^{\bar{0}} t^{(2k+1)\mathbf{m}_2} Q(q^\alpha t^{\mathbf{m}_2}) \right) \cdot v_0 = 0,$$

$$\left(t_0^{\bar{1}} t^{(2k+1)\mathbf{m}_2} Q(t^{\mathbf{m}_2}) - (-1)^{m_{11}+1} q^{(2k+1)\alpha} t_0^{\bar{1}} t^{(2k+1)\mathbf{m}_2} Q(q^\alpha t^{\mathbf{m}_2}) \right) \cdot v_0 = 0, \tag{19}$$

for all $k \in \mathbf{Z}$ and $Q(t^{\mathbf{m}_2}) \in \mathbf{C}[t^{\pm 2\mathbf{m}_2}]$ with $P_o(t^{\mathbf{m}_2}) \mid Q(t^{\mathbf{m}_2})$, where a_Q is the constant term of $t^{2k\mathbf{m}_2} Q(t^{\mathbf{m}_2})$.

Let $P(t^{\mathbf{m}_2}) = \sum_{i=0}^{n} a_i t^{2i\mathbf{m}_2}$ be the product of $P_o(t^{\mathbf{m}_2})$ and $P_e(t^{\mathbf{m}_2})$. We see that both (3.13) and (3.19) hold for $P(t^{\mathbf{m}_2})$. Thus one can directly deduce that both (3.10) and (3.12) hold for $P(t^{\mathbf{m}_2})$ and $v_0 \in V_0$. Since v_0 is a eigenvector of $t_0^{\bar{1}}$, we have

$$0 = t_0^{\bar{1}} \cdot t_0^{\bar{1}} t^{(2k+1)\mathbf{m}_2} P(t^{\mathbf{m}_2}) \cdot v_0 = [t_0^{\bar{1}}, t_0^{\bar{1}} t^{(2k+1)\mathbf{m}_2} P(t^{\mathbf{m}_2})] \cdot v_0 = 2t_0^{\bar{0}} t^{(2k+1)\mathbf{m}_2} P(t^{\mathbf{m}_2}) \cdot v_0,$$

and

$$0 = t_0^{\bar{1}} \cdot t_0^{\bar{1}} t^{(2k+1)\mathbf{m}_2} P(q^{\alpha} t^{\mathbf{m}_2}) \cdot v_0 = [t_0^{\bar{1}}, t_0^{\bar{1}} t^{(2k+1)\mathbf{m}_2} P(q^{\alpha} t^{\mathbf{m}_2})] \cdot v_0 = 2t_0^{\bar{0}} t^{(2k+1)\mathbf{m}_2} P(q^{\alpha} t^{\mathbf{m}_2}) \cdot v_0,$$

which deduces (3.11) for $P(t^{\mathbf{m}_2})$ and v_0.

From the definition of Lie subalgebra L_0, one can easily deduces that if (3.10)–(3.12) hold for any $v \in V$, then they also hold for $t_0^s t^{k\mathbf{m}_2} \cdot v, \ \forall \ s \in \mathbf{Z}/2\mathbf{Z}, \ k \in \mathbf{Z}$. This completes the proof of this direction since V_0 is an irreducible L_0-module.

"$\Longleftarrow$". We first show the following claim by induction on s.

Claim 3. *For any $s \in \mathbf{Z}_+$, there exists a polynomial $P_s(t^{\mathbf{m}_2}) = \sum_{j \in \mathbf{Z}} a_{s,j} t^{2j\mathbf{m}_2} \in \mathbf{C}[t^{2\mathbf{m}_2}]$ such that*

$$\left(t_0^{\bar{0}} t^{2k\mathbf{m}_2} P_s(t^{\mathbf{m}_2}) - q^{2k\alpha} t_0^{\bar{0}} t^{2k\mathbf{m}_2} P_s(q^{\alpha} t^{\mathbf{m}_2}) + a_{s,-k} q^{-4k^2 m_{21} m_{22}} \beta\right) \cdot V_{-s} = 0,$$

$$t_0^{\bar{0}} t^{(2k+1)\mathbf{m}_2} P_s(t^{\mathbf{m}_2}) \cdot V_{-s} = t_0^{\bar{1}} t^{k\mathbf{m}_2} P_s(t^{\mathbf{m}_2}) \cdot V_{-s} = 0,$$

$$t_0^r t^{-\mathbf{m}_1 + k\mathbf{m}_2} P_s(t^{\mathbf{m}_2}) \cdot V_{-s} = 0, \ \ \forall \ r \in \mathbf{Z}_2, k \in \mathbf{Z}.$$

By the assumption and the definition of L_0-module V_0, one can deduce that the claim holds for $s = 0$ with $P_0(t^{\mathbf{m}_2}) = P(t^{\mathbf{m}_2})$. Suppose it holds for s. Let us consider the claim for $s+1$.

The equations in the claim are equivalent to

$$\left(t_0^{\bar{0}} Q(t^{\mathbf{m}_2}) - t_0^{\bar{0}} Q(q^{\alpha} t^{\mathbf{m}_2}) + a_Q \beta\right) \cdot V_{-s} = 0,$$

$$t_0^{\bar{0}} t^{(2k+1)\mathbf{m}_2} Q(t^{\mathbf{m}_2}) \cdot V_{-s} = t_0^{\bar{1}} t^{k\mathbf{m}_2} Q(t^{\mathbf{m}_2}) \cdot V_{-s} = 0,$$

$$t_0^r t^{-\mathbf{m}_1 + k\mathbf{m}_2} Q(t^{\mathbf{m}_2}) \cdot V_{-s} = 0, \ \ \forall \ r \in \mathbf{Z}_2, k \in \mathbf{Z}, \tag{20}$$

for any $Q(t^{\mathbf{m}_2}) \in \mathbf{C}[t^{\pm 2\mathbf{m}_2}]$ with $P_s(t^{\mathbf{m}_2}) \mid Q(t^{\mathbf{m}_2})$, where a_Q is the constant term of $Q(t^{\mathbf{m}_2})$.

Let $P_{s+1}(t^{\mathbf{m}_2}) = P_s(q^{\alpha} t^{\mathbf{m}_2}) P_s(t^{\mathbf{m}_2}) P_s(q^{-\alpha} t^{\mathbf{m}_2})$. For any $p, r \in \mathbf{Z}_2$, $j, k \in \mathbf{Z}$, using induction and by (3.20) we have

$$\left(t_0^{\bar{0}} t^{2k\mathbf{m}_2} P_{s+1}(t^{\mathbf{m}_2}) - q^{2k\alpha} t_0^{\bar{0}} t^{2k\mathbf{m}_2} P_{s+1}(q^{\alpha} t^{\mathbf{m}_2}) + a_{s+1,-k} q^{-4k^2 m_{21} m_{22}} \beta\right) \cdot t_0^p t^{-\mathbf{m}_1 + j\mathbf{m}_2} \cdot V_{-s}$$

$$= \left[t_0^{\bar{0}} t^{2k\mathbf{m}_2} P_{s+1}(t^{\mathbf{m}_2}) - q^{2k\alpha} t_0^{\bar{0}} t^{2k\mathbf{m}_2} P_{s+1}(q^{\alpha} t^{\mathbf{m}_2}) + a_{s+1,-k} q^{-k^2 m_{21} m_{22}} \beta, t_0^p t^{-\mathbf{m}_1 + j\mathbf{m}_2}\right] \cdot V_{-s}$$

$$= q^{2k m_{22}(-m_{11} + j m_{21})} \left(t_0^p t^{-\mathbf{m}_1 + (2k+j)\mathbf{m}_2} \left(P_{s+1}(q^{-\alpha} t^{\mathbf{m}_2}) - 2q^{2k\alpha} P_{s+1}(t^{\mathbf{m}_2}) + q^{4k\alpha} P_{s+1}(q^{\alpha} t^{\mathbf{m}_2})\right)\right) \cdot V_{-s}$$

$$= 0,$$

Thus, by (3.1), we obtain that

$$\left(t_0^{\bar{0}}t^{2k\mathbf{m}_2}P_{s+1}(t^{\mathbf{m}_2}) - q^{2k\alpha}t_0^{\bar{0}}t^{2k\mathbf{m}_2}P_{s+1}(q^\alpha t^{\mathbf{m}_2}) + a_{s+1,-k}q^{-4k^2m_{21}m_{22}}\beta\right)\cdot V_{-s-1} = 0. \qquad (3.21)$$

Similarly, one can prove that

$$t_0^{\bar{0}}t^{(2k+1)\mathbf{m}_2}P_{s+1}(t^{\mathbf{m}_2})\cdot V_{-s-1} = t_0^{\bar{1}}t^{k\mathbf{m}_2}P_{s+1}(t^{\mathbf{m}_2})\cdot V_{-s-1} = 0, \quad \forall\, k\in\mathbf{Z}. \qquad (3.22)$$

This proves the first two equations in the claim for $s+1$.

Using (3.21), (3.22) and induction, we deduce that for any $l,k\in\mathbf{Z}$, $n,r\in\mathbf{Z}_2$,

$$t_0^n t^{\mathbf{m}_1+l\mathbf{m}_2}\cdot t_0^r t^{-\mathbf{m}_1+k\mathbf{m}_2}P_{s+1}(t^{\mathbf{m}_2})\cdot V_{-s-1}$$

$$= [t_0^n t^{\mathbf{m}_1+l\mathbf{m}_2}, t_0^r t^{-\mathbf{m}_1+k\mathbf{m}_2}P_{s+1}(t^{\mathbf{m}_2})]\cdot V_{-s-1} + t_0^r t^{-\mathbf{m}_1+k\mathbf{m}_2}P_{s+1}(t^{\mathbf{m}_2})\cdot t_0^n t^{\mathbf{m}_1+l\mathbf{m}_2}\cdot V_{-s-1}$$

$$= (-1)^{r(m_{11}+lm_{21})}q^{-m_{11}m_{12}+km_{12}m_{21}-lm_{11}m_{22}+lkm_{21}m_{22}}\left(t_0^{n+r}t^{(l+k)\mathbf{m}_2}P_{s+1}(t^{\mathbf{m}_2})\right.$$

$$- (-1)^{(n+r)m_{11}+nk+rl}q^{(k+l)\alpha}t_0^{n+r}t^{(l+k)\mathbf{m}_2}P_{s+1}(q^\alpha t^{\mathbf{m}_2})$$

$$\left. + a_{s+1,i}\delta_{k+l+2i,0}\delta_{r+n,\bar{0}}q^{-(l+k)^2m_{21}m_{22}}\beta\right)\cdot V_{-s-1}$$

$$= 0.$$

Hence, by (3.2),

$$t_0^r t^{-\mathbf{m}_1+k\mathbf{m}_2}P_{s+1}(t^{\mathbf{m}_2})\cdot V_{-s-1} = 0,$$

for all $r\in\mathbf{Z}_2$, $k\in\mathbf{Z}$, which implies the third equation in the claim for $s+1$. Therefore the claim follows by induction.

From the third equation of the claim and (3.1), we see that

$$\dim V_{-s-1} \le 2\deg(P_{s+1}(t^{\mathbf{m}_2}))\cdot\dim V_s, \quad \forall\, s\in\mathbf{Z}_+,$$

where $\deg(P_{s+1}(t^{\mathbf{m}_2}))$ denotes the degree of polynomial $P_{s+1}(t^{\mathbf{m}_2})$. Hence $M^+(V(\underline{\mu},\psi),\mathbf{m}_1,\mathbf{m}_2)$ $\in\mathcal{O}_\mathbf{Z}$. $\square$

Theorem 3.4 *Let m_{21} be an odd integer. Then $M^+(\underline{\mu},\psi,\mathbf{m}_1,\mathbf{m}_2)\in\mathcal{O}_\mathbf{Z}$ if and only if there exist $b_{10},b_{11},\cdots,b_{1s_1},b_{20},b_{21},\cdots,b_{2s_2},\cdots,b_{r0},b_{r1},\cdots,b_{rs_r}\in\mathbf{C}$ and $\alpha_1,\cdots,\alpha_r\in\mathbf{C}^*$ such that for any $i\in\mathbf{Z}^*$, $j\in\mathbf{Z}_2$,*

$$\psi(t_0 t^{2i\mathbf{m}_2}) = \frac{(b_{10}+b_{11}i+\cdots+b_{1s_1}i^{s_1})\alpha_1^i + \cdots + (b_{r0}+b_{r1}i+\cdots,b_{rs_r}i^{s_r})\alpha_r^i}{(1-q^{2i\alpha})q^{2i^2m_{21}m_{22}}},$$

$$\psi(\beta) = b_{10}+b_{20}+\cdots+b_{r0}, \quad and \quad \psi(m_{21}c_1+m_{22}c_2) = 0,$$

where $\alpha = m_{11}m_{22}-m_{21}m_{12}\in\{\pm1\}$.

18

Proof "$\Longrightarrow$". Let $f_i = \psi((1 - q^{2i\alpha})q^{2i^2 m_{21} m_{22}} t_0^{\bar{0}} t^{2i\mathbf{m}_2})$ for $i \in \mathbf{Z}^*$ and $f_0 = \psi(\beta)$. By Lemma 3.3, there exist complex numbers $a_0, a_1, \cdots, a_n$ with $a_0 a_n \neq 0$ such that

$$\sum_{i=0}^{n} a_i q^{-2i^2 m_{21} m_{22}} f_{k+i} = 0, \ \forall\, k \in \mathbf{Z}.$$

Thus, by using a technique in the proof of Theorem 3.2, we can deduce the result as required.

"$\Longleftarrow$". Set

$$Q(x) = \Big(\prod_{i=1}^{r}(x - \alpha_i)^{s_i+1}\Big)\Big(\prod_{j=1}^{\nu}(x - a_j)\Big)\Big(\prod_{j=1}^{\nu}(x - q^{2\alpha}a_j)\Big) =: \sum_{i=1}^{n} b_i x^i,$$

and

$$f_i = \psi\Big((1 - q^{2i\alpha})q^{2i^2 m_{21} m_{22}} t_0^{\bar{0}} t^{2i\mathbf{m}_2}\Big), \ \forall\, i \in \mathbf{Z}^*, \quad f_0 = \psi(\beta).$$

Then one can easily verify that

$$\sum_{i=0}^{n} b_i f_{k+i} = 0, \quad \forall\, k \in \mathbf{Z}. \tag{3.23}$$

Meanwhile, we have $(\prod_{j=1}^{\nu}(x - a_j)) \mid x^k Q(x)$ and $(\prod_{j=1}^{\nu}(x - a_j)) \mid x^k Q(q^{2\alpha}x)$ for any $k \in \mathbf{Z}$, which deduces

$$\sum_{i=1}^{n} b_i q^{\frac{1}{2}(2i+2k+1)^2 m_{22} m_{21}} t_0^s t^{(2i+2k+1)\mathbf{m}_2} \cdot V_0 = 0, \tag{24}$$

$$\sum_{i=1}^{n} b_i q^{2i\alpha} q^{\frac{1}{2}(2i+2k+1)^2 m_{22} m_{21}} t_0^s t^{(2i+2k+1)\mathbf{m}_2} \cdot V_0 = 0, \quad \forall\, s \in \mathbf{Z}_2, \tag{25}$$

and

$$\sum_{i=1}^{n} b_i q^{2(i+k)^2 m_{22} m_{21}} t_0^{\bar{1}} t^{2(i+k)\mathbf{m}_2} \cdot V_0 = 0, \tag{26}$$

$$\sum_{i=1}^{n} b_i q^{2i\alpha} q^{2(i+k)^2 m_{22} m_{21}} t_0^{\bar{1}} t^{2(i+k)\mathbf{m}_2} \cdot V_0 = 0, \tag{27}$$

by Remark 2.6. Let $b_i' = q^{2i^2 m_{21} m_{22}} b_i$ for $0 \leq i \leq n$ and $P(x) = \sum_{i=1}^{n} b_i' x^i$. By (3.23) and the

construction of $V(\underline{\mu}, \psi)$, we have

$$\left(t_0^{\bar 0}t^{2k\mathbf{m_2}}P(t^{2\mathbf{m_2}}) - q^{2k\alpha}t_0^{\bar 0}t^{2k\mathbf{m_2}}P(q^{2\alpha}t^{2\mathbf{m_2}}) + b'_{-k}q^{-4k^2m_{21}m_{22}}\beta\right)\cdot V_0$$

$$= q^{-2k^2m_{21}m_{22}}\psi\Big(\sum_{i=1}^{n}b_i(1-q^{2(k+i)\alpha})q^{2(k+i)^2m_{22}m_{21}}t_0^{\bar 0}t^{2(k+i)\mathbf{m_2}} + b_{-k}\beta\Big)\cdot V_0$$

$$= q^{-2k^2m_{21}m_{22}}\sum_{i=1}^{n}b_if_{k+i}\cdot V_0$$

$$= 0,$$

which deduces (3.10). Similarly, we have, for any $k \in \mathbf{Z}$,

$$t_0^s t^{(2k+1)\mathbf{m_2}}P(t^{2\mathbf{m_2}})\cdot V_0 = \sum_{i=1}^{n}b_iq^{(2i^2+4ki+2i)m_{21}m_{22}}t_0^s t^{(2k+2i+1)\mathbf{m_2}}\cdot V_0$$

$$= q^{-2k^2-2k-\frac{1}{2}}\sum_{i=1}^{n}b_iq^{\frac{1}{2}(2k+2i+1)^2m_{21}m_{22}}t_0^s t^{(2k+2i+1)\mathbf{m_2}}\cdot V_0$$

$$= 0,$$

and

$$t_0^s t^{(2k+1)\mathbf{m_2}}P(q^{2\alpha}t^{2\mathbf{m_2}})\cdot V_0 = \sum_{i=1}^{n}b_iq^{2i\alpha+(2i^2+4ki+2i)m_{21}m_{22}}t_0^s t^{(2k+2i+1)\mathbf{m_2}}\cdot V_0$$

$$= q^{-2k^2-2k-\frac{1}{2}}\sum_{i=1}^{n}b_iq^{2i\alpha}q^{\frac{1}{2}(2k+2i+1)^2m_{21}m_{22}}t_0^s t^{(2k+2i+1)\mathbf{m_2}}\cdot V_0$$

$$= 0,$$

by (3.24) and (3.25) respectively. Now one can easily deduce the following equation

$$t_0^{\bar 1}t^{2k\mathbf{m_2}}P(t^{2\mathbf{m_2}})\cdot V_0 = 0, \quad \text{and} \quad t_0^{\bar 1}t^{2k\mathbf{m_2}}P(q^{2\alpha}t^{2\mathbf{m_2}})\cdot V_0 = 0,$$

by using (3.26) and (3.27) respectively. Therefore (3.10)–(3.12) hold for $P(t^{2\mathbf{m_2}}) = \sum_{i=1}^{n}b'_it^{2i\mathbf{m_2}}$. Thus $M^+(\underline{\mu}, \psi, \mathbf{m_1}, \mathbf{m_2}) \in \mathcal{O}_{\mathbf{Z}}$ by Lemma 3.3. $\qquad\square$

Remark 3.5 *A linear function ψ over L_0 having the form as described in Theorem 3.2 is called an exp-polynomial function over L_0. Similarly, a linear function ψ over $\mathcal{A}$ having the form as described in Theorem 3.4 is called an exp-polynomial function over $\mathcal{A}$.*

§4 Classification of the generalized highest weight irreducible Z-graded L-modules

Lemma 4.1 *If V is a nontrivial irreducible generalized highest weight $\mathbf{Z}$-graded L-module corresponding to a $\mathbf{Z}$-basis $B = \{\mathbf{b_1}, \mathbf{b_2}\}$ of $\mathbf{Z}^2$, then*

(1) *For any $v \in V$ there is some $p \in \mathbf{N}$ such that $t_0^i t^{m_1\mathbf{b_1}+m_2\mathbf{b_2}}\cdot v = 0$ for all $m_1, m_2 \geq p$ and $i \in \mathbf{Z}_2$.*

(2) *For any $0 \neq v \in V$ and $m_1, m_2 > 0$, $i \in \mathbf{Z}_2$, we have $t_0^i t^{-m_1\mathbf{b_1}-m_2\mathbf{b_2}}\cdot v \neq 0$.*

Proof   Assume that v_0 is a generalized highest weight vector corresponding to the **Z**-basis $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ of $\mathbf{Z}^2$.

(1) By the irreducibility of V and the PBW theorem, there exists $u \in U(L)$ such that $v = u \cdot v_0$, where u is a linear combination of elements of the form

$$u_n = (t_0^{k_1} t^{i_1 \mathbf{b}_1 + j_1 \mathbf{b}_2}) \cdot (t_0^{k_2} t^{i_2 \mathbf{b}_1 + j_1 2 \mathbf{b}_2}) \cdots (t_0^{k_n} t^{i_n \mathbf{b}_1 + j_n \mathbf{b}_2}),$$

where, "$\cdot$" denotes the product in $U(L)$. Thus, we may assume $u = u_n$. Take

$$p_1 = -\sum_{i_s < 0} i_s + 1, \quad p_2 = -\sum_{j_s < 0} j_s + 1.$$

By induction on n, one gets that $t_0^k t^{i\mathbf{b}_1 + j\mathbf{b}_2} \cdot v = 0$ for any $k \in \mathbf{Z}_2, i \geq p_1$ and $j \geq p_2$, which gives the result with $p = \max\{p_1, p_2\}$.

(2) Suppose there are $0 \neq v \in V$ and $i \in \mathbf{Z}_2, m_1, m_2 > 0$ with

$$t_0^i t^{-m_1 \mathbf{b}_1 - m_2 \mathbf{b}_2} \cdot v = 0.$$

Let p be as in the proof of (1). Then

$$t_0^i t^{-m_1 \mathbf{b}_1 - m_2 \mathbf{b}_2}, \ t_0^j t^{\mathbf{b}_1 + p(m_1 \mathbf{b}_1 + m_2 \mathbf{b}_2)}, \ t_0^j t^{\mathbf{b}_2 + p(m_1 \mathbf{b}_1 + m_2 \mathbf{b}_2)}, \ \ \forall j \in \mathbf{Z}_2,$$

act trivially on v. Since the above elements generate the Lie algebra L. So V is a trivial module, a contradiction. $\qquad\square$

Lemma 4.2   *If $V \in \mathcal{O}_{\mathbf{Z}}$ is a generalized highest weight L-module corresponding to the **Z**-basis $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ of $\mathbf{Z}^2$, then V must be a highest or lowest weight module.*

Proof   Suppose V is a generalized highest weight module corresponding to the **Z**-basis $\{\mathbf{b}_1 = b_{11}\mathbf{m}_1 + b_{12}\mathbf{m}_2, \ \mathbf{b}_2 = b_{21}\mathbf{m}_1 + b_{22}\mathbf{m}_2\}$ of $\mathbf{Z}^2$. By shifting index of V_i if necessary, we can suppose the highest degree of V is 0. Let $a = b_{11} + b_{21}$ and

$$\wp(V) = \{m \in \mathbf{Z} \mid V_m \neq 0\}.$$

We may assume $a \neq 0$: In fact, if $a = 0$ we can choose $\mathbf{b}_1' = 3\mathbf{b}_1 + \mathbf{b}_2, \ \mathbf{b}_2' = 2\mathbf{b}_1 + \mathbf{b}_2$, then V is a generalized highest weight **Z**-graded module corresponding to the **Z**-basis $\{\mathbf{b}_1', \mathbf{b}_2'\}$ of $\mathbf{Z}^2$. Replacing $\mathbf{b}_1, \mathbf{b}_2$ by $\mathbf{b}_1', \mathbf{b}_2'$ gives $a \neq 0$.

Now we prove that if $a > 0$ then V is a highest weight module. Let

$$\mathcal{A}_i = \{j \in \mathbf{Z} \mid i + aj \in \wp(V)\}, \ \ \forall \ 0 \leq i < a.$$

Then there is $m_i \in \mathbf{Z}$ such that $\mathcal{A}_i = \{j \in \mathbf{Z} \mid j \leq m_i\}$ or $\mathcal{A}_i = \mathbf{Z}$ by Lemma 4.1(2).

Set $\mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2$. We want to prove $\mathcal{A}_i \neq \mathbf{Z}$ for all $0 \leq i < a$. Otherwise, (by shifting the index of $\mathcal{A}_i$ if necesary) we may assume $\mathcal{A}_0 = \mathbf{Z}$. Thus we can choose $0 \neq v_j \in V_{aj}$ for any $j \in \mathbf{Z}$. By Lemma 4.1(1), we know that there is $p_{v_j} > 0$ with

$$t_0^k t^{s_1 \mathbf{b}_1 + s_2 \mathbf{b}_2} \cdot v_j = 0, \ \ \forall \ s_1, s_2 > p_{v_j}, \ k \in \mathbf{Z}/2\mathbf{Z}. \tag{4.1}$$

Choose $\{k_j \in \mathbf{N} \mid j \in \mathbf{N}\}$ and $v_{k_j} \in V_{ak_j}$ such that

$$k_{j+1} > k_j + p_{v_{k_j}} + 2. \tag{4.2}$$

We prove that $\{t_0^{\bar{0}} t^{-k_j \mathbf{b}} \cdot v_{k_j} \mid j \in \mathbf{N}\} \subset V_0$ is a set of linearly independent vectors, from which we can get a contradiction and thus deduces the result as we hope.

Indeed, for any $r \in \mathbf{N}$, there exists $a_r \in \mathbf{N}$ such that $t_0^0 t^{x\mathbf{b} + \mathbf{b}_1} v_{k_r} = 0$, $\forall x \geq a_r$ by Lemma 4.1(1). On the other hand, we know that $t_0^0 t^{x\mathbf{b} + \mathbf{b}_1} \cdot v_{k_r} \neq 0$ for any $x < -1$ by Lemma 4.1(2). Thus we can choose $s_r \geq -2$ such that

$$t_0^{\bar{0}} t^{s_r \mathbf{b} + \mathbf{b}_1} \cdot v_{k_r} \neq 0, \qquad t_0^{\bar{0}} t^{x\mathbf{b} + \mathbf{b}_1} \cdot v_{k_r} = 0, \ \forall x > s_r. \tag{4.3}$$

By (4.2) we have $k_r + s_r - k_j > p_{v_j}$ for all $1 \leq j < r$. Hence by (4.1) we know that for all $1 \leq j < r$,

$$t_0^{\bar{0}} t^{(k_r + s_r)\mathbf{b} + \mathbf{b}_1} \cdot t_0^{\bar{0}} t^{-k_j \mathbf{b}} \cdot v_{k_j}$$
$$= [t_0^{\bar{0}} t^{(k_r + s_r)\mathbf{b} + \mathbf{b}_1}, t_0^{\bar{0}} t^{-k_j \mathbf{b}}] \cdot v_{k_j}$$
$$= q^{-k_j((k_r + s_r)(b'_{12} + b'_{22}) + b'_{12})(b'_{11} + b'_{21})}(1 - q^{k_j(b'_{12}b'_{21} - b'_{11}b'_{22})}) t_0^{\bar{0}} t^{(k_r + s_r - k_j)\mathbf{b} + \mathbf{b}_1} \cdot v_{k_j}$$
$$= 0,$$

where

$$b'_{11} = b_{11}m_{11} + b_{12}m_{21}, \ b'_{12} = b_{11}m_{12} + b_{12}m_{22}, \ b'_{21} = b_{21}m_{11} + b_{22}m_{21}, \ b'_{22} = b_{21}m_{12} + b_{22}m_{22}.$$

Now by (4.2) and (4.3), one gets

$$t_0^{\bar{0}} t^{(k_r + s_r)\mathbf{b} + \mathbf{b}_1} \cdot t_0^{\bar{0}} t^{-k_r \mathbf{b}} \cdot v_{k_r}$$
$$= [t_0^{\bar{0}} t^{(k_r + s_r)\mathbf{b} + \mathbf{b}_1}, t_0^{\bar{0}} t^{-k_r \mathbf{b}}] \cdot v_{k_r}$$
$$= q^{-k_r((k_r + s_r)(b'_{12} + b'_{22}) + b'_{12})(b'_{11} + b'_{21})}(1 - q^{k_r(b'_{12}b'_{21} - b'_{11}b'_{22})}) t_0^{\bar{0}} t^{s_r \mathbf{b} + \mathbf{b}_1} \cdot v_{k_r}$$
$$\neq 0.$$

Hence if $\sum_{j=1}^n \lambda_j t_0^{\bar{0}} t^{-k_j \mathbf{b}} \cdot v_{k_j} = 0$ then $\lambda_n = \lambda_{n-1} = \cdots = \lambda_1 = 0$ by the arbitrariness of r. So we see that $\{t_0^{\bar{0}} t^{-k_j \mathbf{b}} \cdot v_{k_j} \mid j \in \mathbf{N}\} \subset V_0$ is a set of linearly independent vectors, which

contradicts the fact that $V \in \mathcal{O}_{\mathbf{Z}}$. Therefore, for any $0 \leq i < a$, there is $m_i \in \mathbf{Z}$ such that $\mathcal{A}_i = \{j \in \mathbf{Z} \mid j \leq m_i\}$, which deduces that V is a highest weight module since $\wp(V) = \bigcup_{i=0}^{a-1} \mathcal{A}_i$.

Similarly, one can prove that if $a < 0$ then V is a lowest weight module. $\quad\square$

From the above lemma and the results in Section 3, we have the following theorem.

Theorem 4.3 V *is a quasi-finite irreducible $\mathbf{Z}$-graded L-module if and only if one of the following statements hold:*

(1) V *is a uniformly bounded module;*

(2) *If m_{21} is an even integer then there exists an exp-polynomial function ψ over L_0 such that*

$$V \cong M^+(\psi, \mathbf{m}_1, \mathbf{m}_2) \quad or \quad V \cong M^-(\psi, \mathbf{m}_1, \mathbf{m}_2);$$

(3) *If m_{21} is an odd integer then there exist an exp-polynomial function ψ over $\mathcal{A}$, a finite sequence of nonzero distinct numbers $\underline{\mu} = (a_1, \cdots, a_\nu)$ and some finite dimensional irreducible sl_2-modules $V_1, \cdots, V_\nu$ such that*

$$V \cong M^+(\underline{\mu}, \psi, \mathbf{m}_1, \mathbf{m}_2) \quad or \quad V \cong M^-(\underline{\mu}, \psi, \mathbf{m}_1, \mathbf{m}_2).$$

Theorem 4.4 (Main Theorem) *If V is a quasi-finite irreducible $\mathbf{Z}$-graded L-module with nontrivial center then one of the following statements must hold:*

(1) *If m_{21} is an even integer then there exists an exp-polynomial function ψ over L_0 such that*

$$V \cong M^+(\psi, \mathbf{m}_1, \mathbf{m}_2) \quad or \quad V \cong M^-(\psi, \mathbf{m}_1, \mathbf{m}_2);$$

(2) *If m_{21} is an odd integer then there exist an exp-polynomial function ψ over $\mathcal{A}$, a finite sequence of nonzero distinct numbers $\underline{\mu} = (a_1, \cdots, a_\nu)$ and some finite dimensional irreducible sl_2 modules $V_1, \cdots, V_\nu$ such that*

$$V \cong M^+(\underline{\mu}, \psi, \mathbf{m}_1, \mathbf{m}_2) \quad or \quad V \cong M^-(\underline{\mu}, \psi, \mathbf{m}_1, \mathbf{m}_2).$$

Proof By Theorem 4.3, we only need to show that V is not a uniformly bounded module. From the definition of Lie algebra L, we see that $\mathcal{H}_i = \langle t_0^{\bar{0}} t^{k\mathbf{m}_i}, m_{i1}c_1 + m_{i2}c_2 \mid k \in \mathbf{Z}^* \rangle$, $i = 1, 2$ are Heisenberg Lie algebras. As V is a quasi-finite irreducible $\mathbf{Z}$-graded L-module, we deduce that $m_{21}c_1 + m_{22}c_2$ must be zero. Thus, by the assumption, we have that $m_{11}c_1 + m_{12}c_2 \neq 0$ since $\{\mathbf{m}_1, \mathbf{m}_2\}$ is a $\mathbf{Z}$-basis of $\mathbf{Z}^2$. Therefore, V is not a uniformly bounded module by a well-known result about the representation of the Heisenberg Lie algebra. $\quad\square$

We close this section by showing that nontrivial modules $M^+(\psi, \mathbf{m}_1, \mathbf{m}_2)$, $M^+(\underline{\mu}, \psi, \mathbf{m}_1, \mathbf{m}_2)$ are not uniformly bounded and not integrable.

Theorem 4.5 *Nontrivial module $M^+(\psi, \mathbf{m}_1, \mathbf{m}_2)$ or $M^+(\underline{\mu}, \psi, \mathbf{m}_1, \mathbf{m}_2)$ is not uniformly bounded.*

Proof Set $V \cong M^+(\psi, \mathbf{m}_1, \mathbf{m}_2)$ or $V \cong M^+(\underline{\mu}, \psi, \mathbf{m}_1, \mathbf{m}_2)$ and $V = \oplus_{k \in \mathbf{Z}_+} V_{-k}$. Since V is not trivial, there exist $v_0 \in V_0$, $k \in \mathbf{Z}^*$ and $l \in \mathbf{Z}_2$ such that $t_0^l t^{k\mathbf{m}_2} \cdot v_0 \neq 0$. Thus

$$t_0^{\bar{0}} t^{\mathbf{m}_1} \cdot t_0^l t^{-\mathbf{m}_1 + k\mathbf{m}_2} \cdot v_0 = [t_0^{\bar{0}} t^{\mathbf{m}_1}, t_0^l t^{-\mathbf{m}_1 + k\mathbf{m}_2}] v_0$$
$$= ((-1)^{lm_{11}} q^{m_{12}(-m_{11} + km_{21})} - q^{m_{11}(-m_{12} + km_{22})}) t_0^l t^{k\mathbf{m}_2} \cdot v_0 \neq 0,$$

which deduces that $t_0^l t^{-\mathbf{m}_1 + k\mathbf{m}_2} \cdot v_0 \neq 0$.

Next, we prove that if $0 \neq v_{-m} \in V_{-m}$ then $t_0^{\bar{0}} t^{-\mathbf{m}_1} \cdot v_{-m} \neq 0$. Suppose $t_0^{\bar{0}} t^{-\mathbf{m}_1} \cdot v_{-m} = 0$ for some $0 \neq v_{-m} \in V_{-m}$. From the construction of V, we know that $t_0^l t^{(m+1)\mathbf{m}_1 \pm \mathbf{m}_2}$ also act trivially on v_{-m} for any $l \in \mathbf{Z}_2$. Since L is generated by $t_0^{\bar{0}} t^{-\mathbf{m}_1}$, $t_0^l t^{(m+1)\mathbf{m}_1 \pm \mathbf{m}_2}$, $l = \bar{0}, \bar{1}$, we see that V is a trivial module, a contradiction.

Set

$$\mathscr{A} = \{(t_0^{\bar{0}} t^{-\mathbf{m}_1})^j \cdot t_0^l t^{(-n+j)\mathbf{m}_1 + k\mathbf{m}_2} \cdot v_0 \mid 0 \leq j < n\} \subset V_{-n}, \forall\, n \in \mathbf{N}.$$

Now we prove that $\mathscr{A}$ is a set of linear independent vectors. If

$$\sum_{j=0}^{n-1} \lambda_j (t_0^{\bar{0}} t^{-\mathbf{m}_1})^j t_0^l t^{(-n+j)\mathbf{m}_1 + k\mathbf{m}_2} \cdot v_0 = 0,$$

then for any $0 \leq i < n - 1$ we have

$$0 = q^{n(n-i)m_{11}m_{12} - k(n-i)m_{12}m_{21}} t_0^{\bar{0}} t^{(n-i)\mathbf{m}_1} \cdot \sum_{j=0}^{n-1} \lambda_j (t_0^{\bar{0}} t^{-\mathbf{m}_1})^j \cdot t_0^l t^{(-n+j)\mathbf{m}_1 + k\mathbf{m}_2} \cdot v_0$$
$$= \sum_{j=0}^{i} \lambda_j q^{j(n-i)m_{11}m_{12}} ((-1)^{l(n-i)m_{11}} - q^{k(n-i)\alpha})(t_0^{\bar{0}} t^{-\mathbf{m}_1})^j \cdot t_0^l t^{(j-i)\mathbf{m}_1 + k\mathbf{m}_2} \cdot v_0,$$

where $\alpha = m_{11}m_{22} - m_{12}m_{21}$, which deduces $\lambda_0 = \cdots = \lambda_{n-1} = 0$. Hence $\mathscr{A}$ is a set of linear independent vectors in V_{-n} and thus

$$\dim V_{-n} \geq n.$$

Therefore V is not a uniformly bounded module by the arbitrariness of n. $\qquad\square$

In [21], Rao gives a classification of the integrable modules with nonzero center for the core of EALAs coordinatized by quantum tori. We want to prove that the L-modules constructed in this paper are in general not integrable. First we recall the concept of the integrable modules. Let τ be the Lie algebra defined in Section 2. A τ-module V is *integrable* if, for any $v \in V$ and $\mathbf{m} \in \mathbf{Z}^2$, there exist $k_1 = k_1(\mathbf{m}, v), k_2 = k_2(\mathbf{m}, v)$ such that $(E_{12}(t^{\mathbf{m}}))^{k_1} \cdot v = (E_{21}(t^{\mathbf{m}}))^{k_2} \cdot v = 0$. Thus by Proposition 2.1, an L-module V is integrable if, for any $v \in V$ and $\mathbf{m} = (2m_1 + 1, m_2) \in \mathbf{Z}^2$, there exist $k_1 = k_1(\mathbf{m}, v), k_2 = k_2(\mathbf{m}, v)$ such that

$$(t_0^{\bar{0}} t^{\mathbf{m}} + t_0^{\bar{1}} t^{\mathbf{m}})^{k_1} \cdot v = 0 = (t_0^{\bar{0}} t^{\mathbf{m}} - t_0^{\bar{1}} t^{\mathbf{m}})^{k_2} \cdot v = 0.$$

Theorem 4.6 *Nontrivial modules $M^+(\psi, \mathbf{m}_1, \mathbf{m}_2)$ or $M^+(\mu, \psi, \mathbf{m}_1, \mathbf{m}_2)$ is not integrable.*

Proof Set $V \cong M^+(\psi, \mathbf{m}_1, \mathbf{m}_2)$ or $V \cong M^+(\mu, \psi, \mathbf{m}_1, \mathbf{m}_2)$ and $V = \oplus_{k \in \mathbf{Z}_+} V_{-k}$. Choose two positive integers a and b such that $\mathbf{m} = a\mathbf{m}_1 + b\mathbf{m}_2 =: (2k+1, l)$. Let $v_0 \in V_0$ be an eigenvector of $t_0^{\bar{1}}$. Then we have

$$(t_0^{\bar{0}} t^{\mathbf{m}} \pm t_0^{\bar{1}} t^{\mathbf{m}}) \cdot v_0 = 0,$$

by the construction of V. On the other hand, by using the isomorphism φ defined in Proposition 2.1, we have

$$\varphi(t_0^{\bar{0}} t^{\mathbf{m}} + t_0^{\bar{1}} t^{\mathbf{m}}) = 2E_{21}(t_1^{m_1+1} t_2^{m_2}), \quad \varphi(t_0^{\bar{0}} t^{\mathbf{m}} - t_0^{\bar{1}} t^{\mathbf{m}}) = 2q^{-m_2} E_{12}(t_1^{m_1} t_2^{m_2}),$$

and

$$\varphi(t_0^{\bar{0}} t^{-\mathbf{m}} + t_0^{\bar{1}} t^{-\mathbf{m}}) = 2E_{21}(t_1^{-m_1} t_2^{-m_2}), \quad \varphi(t_0^{\bar{0}} t^{-\mathbf{m}} - t_0^{\bar{1}} t^{-\mathbf{m}}) = 2q^{m_2+1} E_{12}(t_1^{-m_1-1} t_2^{-m_2}).$$

Thus, by a well-known result on the sl_2-modules, we see that if V is integrable then

$$t_0^{\bar{1}} \cdot v_0 = 0, \quad (t_0^{\bar{0}} t^{-\mathbf{m}} + t_0^{\bar{1}} t^{-\mathbf{m}}) \cdot v_0 = 0, \quad (t_0^{\bar{0}} t^{-\mathbf{m}} - t_0^{\bar{1}} t^{-\mathbf{m}}) \cdot v_0 = 0.$$

So $t_0^{\bar{0}} t^{-\mathbf{m}}$, $t_0^{\bar{1}} t^{-\mathbf{m}}$ act trivially on v_0. On the other hand, the construction of V shows that $t_0^i t^{2\mathbf{m} \pm \mathbf{m}_1}$, $t_0^i t^{2\mathbf{m} \pm \mathbf{m}_2}$ act trivially on v_0. Thus L acts trivially on v_0 since L is generated by $t_0^{\bar{0}} t^{-\mathbf{m}}$, $t_0^{\bar{1}} t^{-\mathbf{m}}$, $t_0^i t^{2\mathbf{m} \pm \mathbf{m}_1}$, $t_0^i t^{2\mathbf{m} \pm \mathbf{m}_2}$. Hence V is a trivial L-module, a contradiction. $\qquad \square$

§5 Two classes of highest weight $\mathbf{Z}^2$-graded L-modules

In this section, we construct two classes of highest weight quasi-finite irreducible $\mathbf{Z}^2$-graded L-modules. For any highest weight $\mathbf{Z}$-graded L-module $V = \oplus_{k \in \mathbf{Z}_+} V_{-k}$, set $V_{\mathbf{Z}^2} = V \otimes \mathbf{C}[x^{\pm 1}]$. We define the action of the elements of L on $V_{\mathbf{Z}^2}$ as follows

$$t_0^i t^{m\mathbf{m}_1 + n\mathbf{m}_2} \cdot (v \otimes x^r) = (t_0^i t^{m\mathbf{m}_1 + n\mathbf{m}_2} \cdot v) \otimes x^{r+n},$$

where $v \in V$, $i \in \mathbf{Z}_2$, $r, m, n \in \mathbf{Z}$. For any $v_{-k} \in V_{-k}$, we define the degree of $v_{-k} \otimes t^r$ to be $-k\mathbf{m}_1 + r\mathbf{m}_2$. Then one can easily see that $V_{\mathbf{Z}^2}$ becomes a $\mathbf{Z}^2$-graded L-module. Let W be an irreducible $\mathbf{Z}$-graded L_0-submodule of $V_0 \otimes \mathbf{C}[x^{\pm 1}]$. We know that the L-module $V_{\mathbf{Z}^2}$ has a unique maximal proper submodule J_W which intersects trivially with W. Then we have the irreducible $\mathbf{Z}^2$ graded L-module

$$V_{\mathbf{Z}^2} / J_W.$$

Now by Theorem 4.3, we have the following result.

Theorem 5.1 (1) *If m_{21} is an even integer then $M_{\mathbf{Z}^2}^+(\psi, \mathbf{m}_1, \mathbf{m}_2)/J_W$ is a quasi-finite irreducible $\mathbf{Z}^2$-graded L-module for any exp-polynomial function ψ over L_0 and any irreducible $\mathbf{Z}$-graded L_0-submodule W of $V_0 \otimes \mathbf{C}[x^{\pm 1}]$.*

(2) *If m_{21} is an odd integer then $M_{\mathbf{Z}^2}^+(\underline{\mu},\psi,\mathbf{m}_1,\mathbf{m}_2)/J_W$ is a quasi-finite irreducible $\mathbf{Z}^2$-graded L-module for any exp-polynomial function ψ over $\mathcal{A}$, any finite sequence of nonzero distinct numbers $\underline{\mu} = (a_1, \cdots, a_\nu)$, any finite dimensional irreducible sl_2-modules $V_1, \cdots, V_\nu$ and irreducible $\mathbf{Z}$-graded L_0-submodule W of $V(\underline{\mu},\psi) \otimes \mathbf{C}[x^{\pm 1}]$.*

Remark 5.2 Since $V_0 \otimes \mathbf{C}[x^{\pm 1}]$ and $V(\underline{\mu},\psi) \otimes \mathbf{C}[x^{\pm 1}]$ are in general not irreducible L_0-modules, $M_{\mathbf{Z}^2}^+(\psi,\mathbf{m}_1,\mathbf{m}_2)$ and $M_{\mathbf{Z}^2}^+(\underline{\mu},\psi,\mathbf{m}_1,\mathbf{m}_2)$ are in general not irreducible. For example, if m_{21} is an even integer then we can define an exp-polynomial function ψ over L_0 as follows

$$\psi(t_0^j t^{i\mathbf{m}_2}) = \frac{(-1)^i + 1}{(1 - (-1)^j q^{i\alpha}) q^{\frac{1}{2} i^2 m_{21} m_{22}}}, \quad \psi(\beta) = 2, \quad \psi(t_0^{\bar{1}} t^{\mathbf{0}}) = \frac{1}{2}.$$

One can check that $W = v_0 \otimes \mathbf{C}[x^{\pm 2}]$ is an irreducible $\mathbf{Z}$-graded L_0-submodule of $v_0 \otimes \mathbf{C}[x^{\pm 1}]$. Thus the $\mathbf{Z}^2$-graded L-module $M_{\mathbf{Z}^2}^+(\psi,\mathbf{m}_1,\mathbf{m}_2)$ corresponding to this function ψ is not irreducible. Suppose m_{21} be an odd integer. Let V_1 be the three dimensional irreducible sl_2-module with the highest weight vector v_2. Denote $E_{21} \cdot v_2$ and $(E_{21})^2 \cdot v_2$ by v_0, v_{-2} respectively. Then, for $\underline{\mu} = (1)$, the exp-polynomial function $\psi = 0$ over $\mathcal{A}$ and the sl_2-module V_1, one can see that

$$W = \langle v_2 \otimes x^{2k} \mid k \in \mathbf{Z}\rangle \oplus \langle v_{-2} \otimes x^{2k} \mid k \in \mathbf{Z}\rangle \oplus \langle v_0 \otimes x^{2k+1} \mid k \in \mathbf{Z}\rangle,$$

is an irreducible $\mathbf{Z}$-graded L_0-submodule of $V(\underline{\mu},\psi)$. Thus the corresponding Z^2-graded L-module $M_{\mathbf{Z}^2}^+(\underline{\mu},\psi,\mathbf{m}_1,\mathbf{m}_2)$ is not an irreducible module.

References

[1] R. Hoegh-Krohn and B. Torresani, Classification and construction of quasi-simple Lie algebra, J. Funct. Anal. 89 (1990), 106–136.

[2] B. N. Allison, S. Azam, S. Berman and Y. Gao, Pianzola extended affine Lie algebras and their root systems, Mem. Amer. Math. Soc. 126 (1997), No. 605.

[3] S. Berman, Y. Gao and Y. Krylyuk, Quantum tori and the structure of elliptic quasi-simple Lie algebras, J. Funct. Anal. 135 (1996), 339–389.

[4] B. N. Allison, S. Berman, Y. Gao and A. Pianzola, A characterization of affine Kac-Moody Lie algebras, Comm. Math. Phys. 185 (1997), 671–688.

[5] I. B. Frenkel, Representations of Kac-Moody algebras and dual resonance models, Lectures in Appl. Math. 21, Amer. Math. Soc., Province, 1985, 325–353.

[6] Y. Gao and Z. Zeng, Hermitian representations of the extended affine Lie algebra $\widetilde{gl_2(\mathbf{C}_q)}$, Adv. Math. 207 (2006), 244–265.

[7] R. V. Moody, S. E. Rao and T. Yokonuma, Toroidal Lie algebras and vertex representations, Geom. Dedecata 35 (1990), 283–307.

[8] H. Yamada, Extended affine Lie algebras and their vertex representations, Publ. RIMS, Kyoto U. 25 (1989), 587–603.

[9] P. Etingof and I. B. Frenkel, Central extensions of current groups in two dimensions, Comm. Math. Phys. 165 (1994), 429–444.

[10] S. Eswara Rao and R. V. Moody, Vertex representations for n-toroidal Lie algebras and a generalization of the Virasoro algebra, Comm. Math. Phys. 159 (1994), 239–264.

[11] S. Berman and B. Cox, Enveloping algebras and representations of toroidal Lie algebras, Pacfic J. Math. 165 (1996), 339–389.

[12] S. Berman, Y. Gao, Y. Krylyuk and E. Neher, The alternative torus and the structure of elliptic quasi-simple Lie algebras of type A_2, Trans. Amer. Math. Soc. 347 (1995), 4315–4363.

[13] B. N. Allison and Y. Gao, The root system and the core of an extended affine Lie algebra, Selecta Math. (N.S.) 7 (2) (2001), 149–212.

[14] Y. Yoshii, Jodan tori, C. R. Math. Rep. Acad. Sci. Canada 18 (1996), 153–158.

[15] Y. I. Manin, Topics in Noncommutative Geometry, Princeton Univ. Press, 1991.

[16] E. Kirkman, C. Procesi and L. Small, A q-analog for the Virasoro algebra, Comm. Algebra 22 (1994) 3755–3774.

[17] H. P. Jakobsen and V. G. Kac, A new class of unitarizable highest weight representations of infinite-dimensional Lie algebras. II, J. Funct. Anal. 82 (1989), 69–90.

[18] S. Berman and J. Szmigielski, Principal realization for extended affine Lie algebra of type sl_2 with coordinates in a simple quantum torus with two variables, Contemp. Math. 248 (1999), 39–67.

[19] Y. Gao, Vertex operators arising from the homogeneous realization for $\widehat{gl}_N$, Comm. Math. Phys. 211 (2000), 745–777.

[20] Y. Gao, Representations of extended affine Lie algebras coordinatized by certain quantum tori, Composito Math. 123 (2000), 1–25.

[21] S. Eswara Rao, A class of integrable modules for the core of EALA coordinatized by quantum tori, J. Algebra 275 (2004), 59–74.

[22] S. Eswara Rao, Unitary modules for EALAs coordinatized by a quantum torus, Comm. Algebra 31 (2003), 2245–2256.

[23] H. Zhang and K. Zhao, Representations of the Virasoro-like Lie algebras and its q-analog, Comm. Algebra 24 (1996), 4361–4372.

[24] C. Jiang and D. Meng, The derivation algebra of the associative algebra $\mathbf{C}_q[X, Y, X^{-1}, Y^{-1}]$, Comm. Algebra 26 (1998), 1723–1736.

[25] S. Eswara Rao and K. Zhao, Highest weight irreducible representations of rank 2 quantum tori, Math. Res. Lett., 11 (2004), 615–628.

[26] W. Lin and S. Tan, Nonzero level Harish-Chandra modules over the q-analog of the Virasoro-like algebra, J. Algebra 297 (2006), 254–272.

[27] W. Lin and S. Tan, Graded modules for the q-analog Virasoro-like algebra, submitted.

[28] V.G. Kac and A. Radul, Quasi-finite highest weight modules over the Lie algebra of differential operators on the circle, Commun. Math. Phys. 157 (1993), 429–457.

[29] O. Mathieu, Classification of Harish-Chandra modules over the Virasoro algebra, Invent. Math. 107 (1992), 225–234.

[30] S. Eswara Rao, On representations of loop algebras, Comm. Algebra 21 (1993), 2131–2153.

[31] Y. Su, Classification of Harish-Chandra modules over the higher rank Virasoro algebras, Comm. Math. Phys. 240 (2003), 539—551.